Estimation under uncertainties of acoustic and electromagnetic fields from noisy observations

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Abstract

The creation and justification of the methods for minimax estimation of parameters of the external boundary value problems for the Helmholtz equation in unbounded domains are considered. When observations are distributed in subdomains, the determination of minimax estimates is reduced to the solution of integro-differential equations in bounded domains. When observations are distributed on a system of surfaces the problem is reduced to solving integral equations on an unclosed bounded surface which is a union of the boundary of the domain and this system of surfaces. Minimax estimation of the solutions to the boundary value problems from point observations is also studied.

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Introduction

In the system analysis of complex processes described by partial differential equations (PDEs), an important problem is the optimal reconstruction (estimation) of parameters of the equations, like values of some functionals on their solutions or right-hand sides, from observations, which depend on the same solutions.

These problems play an important role in various areas of science and engineering. Depending on the character of a priori information, stochastic or deterministic approaches are possible. The choice is determined by the nature of the problem parameters which can be random or not. Moreover, the optimality of estimations depends on a criterion with respect to which a given value is evaluated.

The field of optimal control of PDEs has been strongly influenced by the work of J.L. Lions, who started the systematic study of optimal control problems for PDEs in [1], in particular, singular perturbation problems in [2] and ill-posed problems in [3]. A possible direction of research in this field consists in extending results from the finite-dimensional case such as Pontryagin's principle, second-order conditions, structure of bang-bang controls, singular arcs and so on. On the other hand partial differential equations have specific features such as finiteness of propagation for hyperbolic systems, or the smoothing effect of parabolic systems, so that they may present qualitative properties that are deeply different from the ones in the finite-dimensional case. The present study is devoted to a class of problems of optimal control and estimation for a specific family of PDEs of mathematical physics.

In practice, the data of boundary value problems (BVPs) for differential equations that simulate a physical or technological object are always given with uncertainty. For example, the right-hand sides of the equations, initial or boundary conditions may be known approximately; that is, they belong to certain bounded sets in the corresponding functional spaces.

For solving the estimation problems we must have supplementary data (observations)

$$y = C\varphi + \eta,$$

where C is an operator that specifies the method of measuring and η is the measurement error. As a rule, this error is not known and belongs to a certain given set and the operator is not invertible. Therefore, in general, from given y, it is not possible to uniquely reconstruct the sought-for solution φ of a BVP and, consequently, quantity $l(\varphi)$, where l is a given linear continuous functional. We see that a natural problem arises: to determine a quantity $\widehat{l(\varphi)}$ which would provide the best (in a certain sense) approximation to the sought-for $l(\varphi)$.

Let us briefly characterize the minimax approach to the solution of this problem. We are looking for linear with respect to observations optimal estimates of functionals of solutions and right-hand sides of BVPs based upon the condition of minimum of the maximal mean square error of estimation taken over the subsets mentioned above.

These estimates were called minimax a priori or minimax program estimates (see [27], [28]).

The situation when the unknown parameters of equations and observations are perturbed by noise whose statistical characteristics are not known completely constitutes the case of special interest.

In the absence of true information about distribution of random perturbations, the minimax approach proved to be a useful solution technique. This approach initiated and developed by N.N. Krasovskii [27], A.B. Kurzhanskii [28], O.G. Nakonechnyi [4], N.F. Kirichenko [16], and B.M. Pshenichnyi enabled one to find optimal estimates of the BVP parameters for ordinary differential equations corresponding to the worst realizations of random perturbations.

The present work is devoted to the creation and rigorous justification of constructive minimax estimation methods of parameters of the external BVPs for the Helmholtz equation in arbitrary unbounded domains with finite boundaries. We reduce the deter-

mination of minimax estimates to the solution of certain integro-differential equations in bounded domains when observations are distributed in subdomains. When observations are distributed on a system of surfaces (that simulate e.g. antennas) the problem is reduced to solving some integral equations on an unclosed bounded surface which is a union of the boundary of the domain and this system of surfaces.

These estimation problem are of tremendous significance in many areas of applied electromagnetics, acoustics, contact mechanics. Therefore, comprehensive theoretical analysis of estimation techniques is an urgent task.

Methods and objectives. The study is aimed at elaboration of the methods of guaranteed estimation of the values of linear functionals defined on solutions to external BVPs for the Helmholtz equation and their right-hand sides.

This task can be fulfilled if the following problems are solved:

- To reduce estimation of the values of functionals defined on the solutions to external BVPs and the right-hand sides of equations that enter the problem statement to certain problems of optimal control of systems that are described by certain conjugate BVPs for the Helmholtz equation in bounded domains with a quadratic quality criterion.
- To obtain, for given restrictions on the unknown second moments of observation noise and unknown deterministic data of the BVPs under study, the systems of integro-differential and integral equations such that the minimax estimates of functionals are expressed in terms of their solutions
- To prove unique solvability of the obtained systems of integro-differential and integral equations.

The object of study is observation problems under uncertainty when the functions that are observed on a system of subdomains or surfaces are coupled with the solutions to the considered BVPs via linear operators with additive measurement errors.

The method of study. The systems of integro-differential and integral equations obtained in this work whose solutions are used to express minimax estimates are based on the theory of generalized solutions to BVPs for the Helmholtz equation, utilization of the so-called Dirichlet-to-Neumann (DtN) data-transforming operators, and the the theory of potential in Sobolev spaces.

A remark on novelty. For the first time we consider the statement of the problem of minimax estimation of the parameters of external BVPs for the Helmholtz equation with general boundary conditions that arise in the mathematical theory of wave diffraction.

For the systems described by such BVPs, we obtain representations for minimax estimates of the values of functionals from the observed solutions and right-hand sides that enter the problem statement; quadratic restrictions are imposed on unknown deterministic data and second moment of observation noise. We also obtain representations for the estimation errors. The representations are obtained in terms of the solutions to certain uniquely solvable systems of integro-differential and integral equations in bounded domains.

When the unknown solutions of the system states are observed that are described by external BVPs for the Helmholtz equation on a system of surfaces, we obtain systems of integro-differential equations in unbounded domains; the required minimax estimates are expressed via the solutions to these systems using integral operators of the potential theory in Sobolev spaces; and the BVPs are reduced to equivalent integral equation systems on multi-connected surfaces (or contours), the latter being a union of the obstacle boundary and the surfaces on which the observations are made.

We prove the unique solvability of the obtained integral equations for any values of the wave number k such that $\operatorname{Im} k \geq 0$, $k \neq 0$.

Practical importance. The estimation techniques elaborated in this work are of big importance for the development of the theory of inverse acoustic and electromagnetic wave scattering by bounded obstacles.

The methods and results of this study may be used for estimating under uncertain conditions of the system states described by BVPs for Helmholtz equation in more complicated domains with the boundaries that stretch to infinity (for example, in a domain $K \setminus \bar{\Omega}$ where K is a layer between two parallel planes and Ω is a bounded domain). In general, the developed estimation methods can be applied to obtaining minimax estimates of parameters for a wide class of problems of mathematical physics.

PART 1

Minimax estimation of the solutions to the Helmholtz problems from observations distributed in subdomains

1.1 Notations and definitions

Let us introduce the notations and definitions that will be used in this work.

$x = (x_1, \ldots, x_n)$ denotes a spatial variable that is varied in an open domain $D \subset \mathbb{R}^n$;

$dx = dx_1 \ldots dx_n$ is a Lebesgue measure in $\mathbb{R}^n$;

$\chi(M)$ is a characteristic function of the set $M \subset \mathbb{R}^n$;

$H^s(\mathbb{R}^n)$ is a Sobolev space of index s:

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) :\ (1 + |y|^2)^{s/2} \mathcal{F}u(y) \in L^2(\mathbb{R}^n)\},$$

where $s \geq 0$, $L^2(\mathbb{R}^n)$ is a space of square integrable functions in $\mathbb{R}^2$ and $\mathcal{F}u(y)$ denotes the Fourier transform of function $u(x)$. If $s < 0$, then $H^{-s}(\mathbb{R}^2)$ denotes the space dual to $H^s(\mathbb{R}^2)$. Let D be a domain in $\mathbb{R}^2$ (not necessarily bounded) with the Lipschitz boundary ∂D. Then $d\partial D$ denotes the element of measure on contour ∂D. $L^2(\partial D)$ is a space of square integrable functions on ∂D; the function space

$$L^2_{\text{loc}}(D) = \big\{u \in \mathcal{D}'(D) :\ u|_{D\cap\Omega_R} \in L^2(D \cap \Omega_R) \quad \text{for every } R > 0 \text{ such that } D \cap \Omega_R \neq \emptyset\big\},$$

Introduce also the Sobolev spaces with the corresponding norms:

$$H^s(D) = \{u|_D :\ u \in H^s(\mathbb{R}^2)\} \ (s \in \mathbb{R}),$$

$$H^s(\partial D) = \begin{cases} u|_{\partial D} :\ u \in H^{s+1/2}(\mathbb{R}^2) & (s > 0), \\ L^2(\partial D) & (s = 0), \\ H^{-s}(\partial D)' \,(\text{dual space with respect to } H^{-s}(\Gamma)) & (s < 0) \end{cases},$$

$$H^1_{\text{loc}}(D) = \big\{u \in \mathcal{D}'(D) :\ u|_{D\cap\Omega_R} \in H^1(D \cap \Omega_R) \quad \text{for every } R > 0 \text{ such that } D \cap \Omega_R \neq \emptyset\big\},$$

$$H^1(D, \Delta) := \{u \in D'(D),\ u \in H^1(D),\ \Delta u \in L^2(D)\}, \tag{1.1}$$

$$H^1_{\mathrm{loc}}(D,\Delta) := \{u \in D'(D),\ u \in H^1_{\mathrm{loc}}(D),\ \Delta u \in L^2_{\mathrm{loc}}(D)\}, \tag{1.2}$$

$$\begin{aligned} H^s_{\mathrm{comp}}(D) := \{u:\ & u \in H^s(D), \\ & u \text{ is identically zero outside some ball centered at the origin}\}, \end{aligned} \tag{1.3}$$

where $\mathcal{D}'(D)$ is the space of distributions in D; here and below by Ω_R we denote the ball $\Omega_R := \{x : |x| < R\}$; the Laplacian is taken in the sense of distributions in D; and $s \in \mathbb{R}$.

Theorem (The trace theorem for $H^1(D)$, [5], p. 102).

For any Lipschitz domain D an operator $\gamma_D : C^0(\bar{D}) \to C^0(\Gamma)$ can be extended to a continuous and surjective operator $\gamma_D : H^1_{\mathrm{loc}}(D) \to H^{1/2}(\partial D)$.

We denote by γ_N the Neumann trace operator

$$(\gamma_N u)(x) := (\operatorname{grad} u(x), n(x))_{\mathbb{R}^n},\ x \in \partial D,\ u \in C^1(\bar{D}),$$

$\gamma_N : C^1(\bar{D}) \to C^0(\partial D)$ and can be extended to a continuous and surjective operator $\gamma_N : H^1_{\mathrm{loc}}(D,\Delta) \to H^{-1/2}(\partial D)$. This operator will further be denoted by $\partial/\partial\nu$.

$< \cdot,\cdot >_{H^{-1/2}(\partial D)\times H^{1/2}(\partial D)}$ denotes the duality relation between spaces $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$, which is an extension of the inner product in $L^2(\partial D)$ in the following sense: if $r \in L^2(\partial D)$, then the following relation holds

$$< r, w >_{H^{-1/2}(\partial D)\times H^{1/2}(\partial D)} = \int_{\partial D} r\bar{w}\, d\partial D \quad \forall w \in H^{1/2}(\partial D). \tag{$*$}$$

Let H be a Hilbert space over the set of complex numbers $\mathbb{C}$ with the inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. By $L^2(\Sigma, H)$ we denote the Bochner space composed of random[1] elements $\xi = \xi(\omega)$ defined on a certain probability space $(\Sigma, \mathcal{B}, P)$ with values in H such that

$$\|\xi\|^2_{L^2(\Sigma,H)} = \int_\Sigma \|\xi(\omega)\|^2_H dP(\omega) < \infty. \tag{1.4}$$

In this case there exists the Bochner integral $\mathbb{E}\xi := \int_\Sigma \xi(\omega)\, dP(\omega) \in H$ which is called the mathematical expectation or the mean value of random element $\xi(\omega)$ and satisfies the condition

$$(h, \mathbb{E}\xi)_H = \int_\Sigma (h, \xi(\omega))_H\, dP(\omega) \quad \forall h \in H. \tag{1.5}$$

[1] Random element ξ with values in Hilbert space H is considered as a function $\xi : \Sigma \to H$ imaging random events $E \in \mathcal{B}$ to Borel sets in H (Borel σ-algebra in H is generated by open sets in H).

Being applied to random variable ξ this expression leads to a usual definition (value) of its mathematical expectation because the Bochner integral (1.4) reduces to a Lebesgue integral with probability measure $dP(\omega)$.

In $L^2(\Sigma, H)$ one can introduce the inner product

$$(\xi, \eta)_{L^2(\Sigma,H)} := \int_\Sigma (\xi(\omega), \eta(\omega))_H \, dP(\omega) \quad \forall \xi, \eta \in L^2(\Sigma, H). \tag{1.6}$$

Applying the sign of mathematical expectation, one can write relationships (1.4)–(1.6) as

$$\|\xi\|^2_{L^2(\Sigma,H)} = \mathbb{E}\|\xi(\omega)\|^2_H, \tag{1.7}$$

$$(h, \mathbb{E}\xi)_H = \mathbb{E}(h, \xi(\omega))_H \quad \forall h \in H, \tag{1.8}$$

$$(\xi, \eta)_{L^2(\Sigma,H)} := \mathbb{E}(\xi(\omega), \eta(\omega))_H \quad \forall \xi, \eta \in L^2(\Sigma, H). \tag{1.9}$$

$L^2(\Sigma, H)$ equipped with norm (1.7) and inner product (1.9) is a Hilbert space.

Consider the problem of finding a solution to the exterior Neumann problem for the Helmholtz equation.

Assume that $\Omega \in \mathbb{R}^2$ is a bounded domain such that $\partial\Omega = \Gamma$ is a Lipschitz contour and Ω_0 is a bounded subdomain of $\mathbb{R}^2 \setminus \bar{\Omega}$, $\bar{\Omega}_0 \subset \mathbb{R}^2 \setminus \bar{\Omega}$. Given a function f defined in the domain $\mathbb{R}^2 \setminus \bar{\Omega}$ such as $f = 0$ outside Ω_0, $f \in L^2(\Omega_0)$ and a function $g \in H^{-1/2}(\Gamma)$, find $\psi \in H^1_{\mathrm{loc}}((\mathbb{R}^2 \setminus \bar{\Omega}), \Delta)$, such that

$$-(\Delta + k^2)\psi(x) = f(x) \quad \text{in} \quad \mathbb{R}^2 \setminus \bar{\Omega}, \tag{1.10}$$

$$\frac{\partial \psi}{\partial \nu} = g \text{ on } \Gamma, \tag{1.11}$$

$$\frac{\partial \psi}{\partial r} - ik\psi = o(1/r^{1/2}), \; r = |x|, \; r \to \infty \tag{1.12}$$

with an equivalent variation formulation: find $\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \bar{\Omega})$ such that

$$\int_{\mathbb{R}^2 \setminus \bar{\Omega}} (\nabla\psi \nabla\bar{\theta} - k^2 \psi \, \bar{\theta}) \, dx = \int_{\Omega_0} f\bar{\theta} \, dx + \int_\Gamma g\bar{\theta} \, d\Gamma \tag{1.13}$$

for all $\theta \in H^1_{\mathrm{comp}}(\mathbb{R}^2 \setminus \bar{\Omega})$ such that ψ satisfies the Sommerfeld radiation condition (1.12). Here we suppose that k is the wave number with $\operatorname{Im} k \geq 0$, $k \neq 0$.

Consider also the following problem: find $\psi \in H^1((\Omega_R \setminus \bar{\Omega}), \Delta)$ such that

$$-(\Delta + k^2)\psi(x) = f(x) \text{ in } \Omega_R \setminus \bar{\Omega}, \tag{1.14}$$

$$\frac{\partial \psi}{\partial \nu} = g \text{ on } \Gamma, \tag{1.15}$$

$$\frac{\partial \psi}{\partial \nu} = M_k^{(1)} \psi \text{ on } \Gamma_R, \tag{1.16}$$

where $f \in L^2(\Omega_0)$ and $g \in H^{-1/2}(\Gamma)$, $M_k^{(1)} : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$ is the Dirichlet-to-Neumann map (DtN map) defined by

$$(M_k^{(1)} \psi)(R, \theta) := \frac{k}{2\pi} \sum_{n \in \mathbb{Z}} \frac{H_n^{(1)\prime}(kR)}{H_n^{(1)}(kR)} \int_0^{2\pi} \psi(R, \phi) e^{in(\theta - \phi)} d\,\phi, \tag{1.17}$$

and Ω_R is a large disk containing $\bar{\Omega}$ and supp f. It is known that problems (1.10)$-$(1.12) and (1.14)$-$(1.16) are equivalent in the following sense (see [6]$-$[11]). If $\psi \in H^1_{\mathrm{loc}}((\mathbb{R}^2 \setminus \bar{\Omega}), \Delta)$, is a solution of (1.10)$-$(1.12), then the restriction of ψ to $\Omega_R \setminus \bar{\Omega}$ belongs to $H^1((\Omega_R \setminus \bar{\Omega}), \Delta)$ and is a solution to (1.14)$-$(1.16). Conversely, if $\psi \in H^1((\Omega_R \setminus \bar{\Omega}), \Delta)$ is a solution to (1.14)$-$(1.16), then this solution extended to the domain $\mathbb{R}^2 \setminus \bar{\Omega}_R$ by

$$\psi(r_P, \theta_P) = \frac{k}{2\pi} \sum_{n \in \mathbb{Z}} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(kR)} \int_0^{2\pi} \psi_0(R, \phi) e^{in(\theta_P - \phi)} d\,\phi, \quad r_P \geq R, \tag{1.18}$$

belongs to $H^1_{\mathrm{loc}}((\mathbb{R}^2 \setminus \bar{\Omega}), \Delta)$ and satisfies (1.10)$-$(1.12). Here $\psi_0(R, \phi) := \psi(R, \phi)$ is the trace of the solution to problem (1.14)$-$(1.16) on Γ_R and (r_P, θ_P) are polar coordinates of the point $P \in \mathbb{R}^2 \setminus \bar{\Omega}_R$.

To formulate an equivalent variational setting of problem (1.14)$-$(1.16), we introduce the continuous sesquilinear form $a(\cdot, \cdot) : H^1(\Omega_R \setminus \bar{\Omega}) \times H^1(\Omega_R \setminus \bar{\Omega}) \to \mathbb{C}$ defined as

$$a(\psi, \theta) := \int_{\Omega_R \setminus \bar{\Omega}} (\nabla \psi \nabla \bar{\theta} - k^2 \psi\, \bar{\theta})\, dx - \int_{\Gamma_R} M_k^{(1)} \psi\, \bar{\theta}\, d\Gamma_R. \tag{1.19}$$

Then an equivalent variational formulation of problem (1.14)$-$(1.16) can be written as follows: find $\psi \in H^1(\Omega_R \setminus \bar{\Omega})$ such that

$$a(\psi, \theta) = l(\theta) \quad \forall \theta \in H^1(\Omega_R \setminus \bar{\Omega}), \tag{1.20}$$

where

$$l(\theta) := \int_{\Omega_0} f \bar{\theta}\, dx + \int_{\Gamma} g \bar{\theta}\, d\Gamma \tag{1.21}$$

is a continuous semilinear functional on $H^1(\Omega_R \setminus \bar{\Omega})$.

In order to set an adjoint problem of (1.14)$-$(1.16) which will be used below we introduce a sesquilinear form

$$a^*(\psi, \theta) := \int_{\Omega_R \setminus \bar{\Omega}} (\nabla \psi \nabla \bar{\theta} - \bar{k}^2 \psi\, \bar{\theta})\, dx - \int_{\Gamma_R} M_{\bar{k}}^{(2)} \psi\, \bar{\theta}\, d\Gamma_R, \tag{1.22}$$

where $M_{\bar{k}}^{(2)} : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$ is the map defined by

$$(M_{\bar{k}}^{(2)} u)(R,\theta) := \frac{\bar{k}}{2\pi} \sum_{z \in \mathbb{Z}} \frac{H_n^{(2)\prime}(\bar{k}R)}{H_n^{(2)}(\bar{k}R)} \int_0^{2\pi} u(R,\phi) e^{in(\theta-\phi)} d\,\phi. \tag{1.23}$$

Lemma 1.1. *The sesquilinear form $a^*(\psi,\theta)$ is an adjoint of $a(\psi,\theta)$.*

Proof. Defining

$$a_1(\psi,\theta) := \int_{\Omega_R \setminus \bar{\Omega}} (\nabla\psi\nabla\overline{\theta} - k^2\psi\,\overline{\theta})\,dx \quad \forall \psi,\theta \in H^1(\Omega_R \setminus \bar{\Omega})$$

and

$$a_2(\psi,\theta) := \int_{\Gamma_R} M_k^{(1)}\psi\,\overline{\theta}\,d\Gamma_R \quad \forall \psi,\theta \in H^1(\Omega_R \setminus \bar{\Omega})$$

we have

$$a(\psi,\theta) = a_1(\psi,\theta) - a_2(\psi,\theta). \tag{1.24}$$

Obviously,

$$a_1^*(\psi,\theta) := \overline{a_1(\theta,\psi)} = \overline{\int_{\Omega_R \setminus \bar{\Omega}} (\nabla\theta\nabla\overline{\psi} - k^2\theta\,\overline{\psi})\,dx} = \int_{\Omega_R \setminus \bar{\Omega}} (\nabla\psi\nabla\overline{\theta} - \bar{k}^2\psi\,\overline{\theta})\,dx. \tag{1.25}$$

Taking into account that

$$\overline{H_n^{(1)}(kR)} = H_n^{(2)}(\bar{k}R), \quad \overline{H_n^{(1)\prime}(kR)} = H_n^{(2)\prime}(\bar{k}R),$$

we find

$$\begin{aligned}
a_2^*(\psi,\theta) &:= \overline{a_2(\theta,\psi)} = \overline{\int_{\Gamma_R} M_k^{(1)}\theta\,\overline{\psi}\,d\Gamma_R} = \int_{\Gamma_R} \overline{M_k^{(1)}\theta}\,\psi\,d\Gamma_R \\
&= R\int_0^{2\pi} \overline{\frac{k}{2\pi}\sum_{n\in\mathbb{Z}} \frac{H_n^{(1)\prime}(kR)}{H_n^{(1)}(kR)} \int_0^{2\pi} \theta(R,\alpha)e^{in(\chi-\alpha)}\,d\alpha}\,\psi(R,\chi)\,d\chi \\
&= R\int_0^{2\pi} \frac{\bar{k}}{2\pi}\sum_{n\in\mathbb{Z}} \frac{\overline{H_n^{(1)\prime}(kR)}}{\overline{H_n^{(1)}(kR)}} \int_0^{2\pi} \overline{\theta(R,\alpha)}e^{-in(\chi-\alpha)}\,d\alpha\,\psi(R,\chi)\,d\chi \\
&= R\int_0^{2\pi} \frac{\bar{k}}{2\pi}\sum_{n\in\mathbb{Z}} \frac{H_n^{(2)\prime}(\bar{k}R)}{H_n^{(2)}(\bar{k}R)} \int_0^{2\pi} \overline{\theta(R,\alpha)}e^{-in(\chi-\alpha)}\,d\alpha\,\psi(R,\chi)\,d\chi \\
&= R\int_0^{2\pi} \frac{\bar{k}}{2\pi}\sum_{n\in\mathbb{Z}} \frac{H_n^{(2)\prime}(\bar{k}R)}{H_n^{(2)}(\bar{k}R)} \int_0^{2\pi} \psi(R,\chi)e^{in(\alpha-\chi)}\,d\chi\,\overline{\theta(R,\alpha)}\,d\alpha = \int_{\Gamma_R} M_{\bar{k}}^{(2)}\psi\,\overline{\theta}\,d\Gamma_R.
\end{aligned} \tag{1.26}$$

From (1.24)−(1.26) and the equality

$$a^*(\psi,\theta) = \overline{a(\theta,\psi)} = a_1^*(\psi,\theta) - a_2^*(\psi,\theta),$$

we obtain the required assertion.

$\square$

Now we can state the variational problem adjoint of (1.20):

Given functions f and g introduced on page 9, find $\psi \in H^1(\Omega_R \setminus \Omega)$ such that

$$a^*(\psi,\theta) = l(\theta) \quad \forall \theta \in H^1(\Omega_R \setminus \bar{\Omega}), \tag{1.27}$$

where functional $l(\theta)$ is defined by (1.21).

It is easy to see that variational problem (1.27) as well as the following problems:

(i) find $\psi \in H^1((\Omega_R \setminus \bar{\Omega}), \Delta)$ such that

$$-(\Delta + \bar{k}^2)\psi(x) = f(x) \text{ in } \Omega_R \setminus \bar{\Omega}, \tag{1.28}$$

$$\frac{\partial \psi}{\partial \nu} = g \text{ on } \Gamma, \tag{1.29}$$

$$\frac{\partial \psi}{\partial \nu} = M_{\bar{k}}^{(2)} \psi \text{ on } \Gamma_R, \tag{1.30}$$

and (ii) find $\psi \in H^1_{\text{loc}}((\mathbb{R}^2 \setminus \bar{\Omega}), \Delta)$ such that

$$-(\Delta + \bar{k}^2)\psi(x) = f(x) \text{ in } \mathbb{R}^2 \setminus \bar{\Omega}, \tag{1.31}$$

$$\frac{\partial \psi}{\partial \nu} = g \text{ on } \Gamma, \tag{1.32}$$

$$\frac{\partial \psi}{\partial r} + i\bar{k}\psi = o(1/r^{1/2}),\ r = |x|,\ r \to \infty,\ \text{если } \operatorname{Im} k \geq 0 \tag{1.33}$$

are equivalent and for any R there exists a positive constant $\alpha > 0$ independent of f and g (but dependent on R) such that

$$\|\psi\|_{H^1(\Omega_R \setminus \bar{\Omega})} \leq \alpha(\|f\|_{H^{-1}(\Omega_0)} + \|g\|_{H^{-1/2}(\Gamma)}). \tag{1.34}$$

If any of the data in BVPs (1.14)−(1.16) or (1.28)−(1.30) is random (e.g. forcing function f or Neumann boundary data g), then the solution ψ will be a random function and corresponding stochastic BVPs are formulated as follows.

Given $f \in L^2(\Sigma, L^2(\Omega_0))$, $g \in L^2(\Sigma, H^{-1/2}(\Gamma))$, find $\psi \in L^2(\Sigma, H^1(\Omega_R))$ such that

$$\mathbb{E}a(\psi,\theta) = \mathbb{E}l(\theta) \quad \forall \theta \in L^2(\Sigma, H^1(\Omega_R \setminus \bar{\Omega})), \tag{1.35}$$

and

$$\mathbb{E}a^*(\psi,\theta) = \mathbb{E}l(\theta) \quad \forall\theta \in L^2(\Sigma, H^1(\Omega_R \setminus \bar{\Omega})), \tag{1.36}$$

where

$$\mathbb{E}l(\theta) := \mathbb{E}\left\{\int_{\Omega_0} f(x,\omega)\overline{\theta(x,\omega)}\,dx + \int_\Gamma g(\cdot,\omega)\overline{\theta(\cdot,\omega)}\,d\Gamma\right\} \tag{1.37}$$

is a continuous semilinear functional on $L^2(\Sigma, L^2(H^1(\Omega_R \setminus \bar{\Omega}))$, $\mathbb{E}a(\cdot,\cdot)$ and $\mathbb{E}a^*(\cdot,\cdot)$: $L^2(\Sigma, L^2(H^1(\Omega_R \setminus \bar{\Omega})) \times L^2(\Sigma, L^2(H^1(\Omega_R \setminus \bar{\Omega})) \to \mathbb{C}$ are continuous sesquilinear forms defined as

$$\begin{aligned}\mathbb{E}a(\psi,\theta) := \mathbb{E}\Bigg\{\int_{\Omega_R\setminus\bar{\Omega}} &(\nabla\psi(x,\omega)\nabla\overline{\theta(x,\omega)} - k^2\psi(x,\omega)\,\overline{\theta}(x,\omega)\,dx \\ &- \int_{\Gamma_R} M_k^{(1)}\psi(\cdot,\omega)\,\overline{\theta(\cdot,\omega)}\,d\Gamma_R\Bigg\}\end{aligned} \tag{1.38}$$

and

$$\begin{aligned}\mathbb{E}a^*(\psi,\theta) := \mathbb{E}\Bigg\{\int_{\Omega_R\setminus\bar{\Omega}} &(\nabla\psi(x,\omega)\nabla\overline{\theta(x,\omega)} - \bar{k}^2\psi(x,\omega)\,\overline{\theta(x,\omega)}\,dx \\ &- \int_{\Gamma_R} M_{\bar{k}}^{(2)}\psi(\cdot,\omega)\,\overline{\theta(\cdot,\omega)}\,d\Gamma_R\Bigg\}.\end{aligned} \tag{1.39}$$

It is known that problems (1.35) and (1.36) have unique solutions and there exists a positive constant $\alpha > 0$ independent of f and g such that

$$\|\psi\|_{L^2(\Sigma,H^1(\Omega_R))} \leq \alpha(\|f\|_{L^2(\Sigma,L^2(\Omega_0))} + \|g\|_{L^2(\Sigma,H^{-1/2}(\Gamma))}). \tag{1.40}$$

Such problems were investigated in [12] and [13], including the construction of finite element methods of their numerical solution.

Problem (1.35) is equivalent to the following ones: find $\psi \in L^2(\Sigma, H^1((\Omega_R \setminus \bar{\Omega}), \Delta))$ such that

$$-(\Delta + k^2)\psi(x,\omega) = f(x,\omega) \text{ in } \Omega_R \setminus \bar{\Omega}, \tag{1.41}$$

$$\frac{\partial\psi(\cdot,\omega)}{\partial\nu} = g(\cdot,\omega) \text{ on } \Gamma, \tag{1.42}$$

$$\frac{\partial\psi(\cdot,\omega)}{\partial\nu} = M_k^{(1)}\psi(\cdot,\omega) \text{ on } \Gamma_R, \tag{1.43}$$

or find $\psi \in L^2(\Sigma, H^1(\Omega_R \setminus \bar{\Omega}))$ satisfying

$$a(\psi(\cdot,\omega),\theta) = l(\theta) \quad \forall\theta \in H^1(\Omega_R \setminus \bar{\Omega}). \tag{1.44}$$

Analogously, problem (1.36) is equivalent to the following problems: find $\psi \in L^2(\Sigma, H^1((\Omega_R \setminus \bar{\Omega}), \Delta))$ such that

$$-(\Delta + \bar{k}^2)\psi(x,\omega) = f(x,\omega) \text{ in } \Omega_R \setminus \bar{\Omega}, \tag{1.45}$$

$$\frac{\partial \psi(\cdot,\omega)}{\partial \nu} = g(\cdot,\omega) \text{ on } \Gamma, \tag{1.46}$$

$$\frac{\partial \psi(\cdot,\omega)}{\partial \nu} = M_{\bar{k}}^{(2)} \psi(\cdot,\omega) \text{ on } \Gamma_R, \tag{1.47}$$

or find $\psi \in L^2(\Sigma, H^1(\Omega_R \setminus \bar{\Omega}))$ satisfying

$$a^*(\psi(\cdot,\omega),\theta) = l(\theta) \quad \forall \theta \in H^1(\Omega_R \setminus \bar{\Omega}). \tag{1.48}$$

The right-hand sides in (1.41)–(1.48) are considered for every realization of random fields $f(\cdot,\omega)$ and $g(\cdot,\omega)$ which belong with probability 1 to the spaces $L^2(\Omega_0)$ and $L^2(\Gamma)$, respectively, and the equalities are satisfied almost certainly.

1.2 Statement of the estimation problem

Consider the exterior Neumann problem for the Helmholtz equation: find a distribution $\varphi \in \mathcal{D}'(\mathbb{R}^2 \setminus \bar{\Omega})$ such that

$$\varphi \in H^1_{\text{loc}}((\mathbb{R}^2 \setminus \bar{\Omega}), \Delta), \tag{1.49}$$

$$-(\Delta + k^2)\varphi(x) = f(x) \quad \text{in} \quad \mathbb{R}^2 \setminus \bar{\Omega}, \tag{1.50}$$

$$\frac{\partial \varphi}{\partial \nu} = g \text{ on } \Gamma, \tag{1.51}$$

$$\frac{\partial \varphi}{\partial r} - ik\varphi = o(1/r^{1/2}),\ r = |x|,\ r \to \infty,\ \operatorname{Im} k \geq 0, \tag{1.52}$$

where k is the wave number with $\operatorname{Im} k \geq 0$, f is a source term distributed in bounded subdomain[2] Ω_0 in $\mathbb{R}^2 \setminus \bar{\Omega}$, $f \in L^2(\Omega_0)$, and $g \in L^2(\Gamma)$.

BVP (1.49)–(1.52) simulates, in particular, acoustic or electromagnetic scattering from an infinite sound-hard (perfectly conducting) cylinder with cross-section Ω.

Denote by G_0 the set of pairs of functions $(\tilde{f}, \tilde{g})$ satisfying the inequality

$$\int_{\Omega_0} Q_1(\tilde{f} - f_0)(x)\overline{(\tilde{f} - f_0)}(x)\,dx + \int_\Gamma Q_2(\tilde{g} - g_0)\overline{(\tilde{g} - g_0)}\,d\Gamma \leq 1, \tag{1.53}$$

and by G_1 the set of random functions $\tilde{\xi}(\cdot) = (\tilde{\xi}_1(\cdot), \ldots, \tilde{\xi}_m(\cdot))$ defined on $\Omega_1 \times \cdots \times \Omega_k$ with integrable second moments $\mathbf{E}|\tilde{\xi}_k(x)|^2$ satisfying conditions

$$\mathbf{E}\tilde{\xi}_k(x) = 0,\ k = \overline{1,m}. \tag{1.54}$$

[2]This means that the function f is defined in the domain $\mathbb{R}^2 \setminus \bar{\Omega}$, $f = 0$ outside Ω_0, and $f \in L^2(\Omega_0)$.

$$\sum_{k=1}^{m}\int_{\Omega_k}\mathbf{E}|\tilde{\xi}_k(x)|^2 r_k^2(x)\,dx \le 1, \tag{1.55}$$

where f_0 is defined in the domain $\mathbb{R}^2\backslash\bar{\Omega}$, $f_0 = 0$ outside Ω_0, $f_0 \in L^2(\Omega_0)$ and $g_0 \in L^2(\Gamma)$, f_0 and g_0 are prescribed functions, Q_1 and Q_2 are Hermitian operators in $L^2(\Omega_0)$ and $L^2(\Gamma)$, respectively, for which there exist bounded inverse operators Q_1^{-1} and Q_2^{-1}, and $\tilde{r}_k(x),\ k=\overline{1,m}$, are nonvanishing functions continuous on sets $\bar{\Omega}_k$.

We suppose that functions $f(x)$ and $g(x)$ in equations (1.50) and (1.51) are not known exactly; it is known only that $(f,g)\in G_0$.

Assume that in subdomains Ω_k, $k=\overline{1,m}$, of domain Ω the following functions are observed

$$y_k(x) = \int_{\Omega_k} g_k(x,y)\varphi(y)\,dy + \xi_k(x,\omega), \quad x\in\Omega_k, \quad k=\overline{1,m}, \tag{1.56}$$

where φ is a solution of BVP (1.49)–(1.52), $g_k(x,y)\in L^2(\Omega_k\times\Omega_k)$ are prescribed functions and $\xi_k(x,\omega)$ are the choice functions of random fields $\xi_k(x)$ with unknown second moments such that $\xi(\cdot) = (\xi_1(\cdot),\ldots,\xi_m(\cdot))\in G_1$.

Let l_0 be a given function defined in a bounded subdomain $\omega_0\subset\Omega$ belonging to $L^2(\omega_0)$.

The estimation problem consists in the following. From observations (1.56) of the state $\varphi(x)$ of the system described by BVP (1.49)–(1.52) under conditions (1.54)–(1.55) it is necessary to estimate the value of the linear functional

$$l(\varphi) = \int_{\omega_0}\overline{l_0(x)}\varphi(x)\,dx \tag{1.57}$$

in the class of the estimates linear with respect to observations which have the form

$$\widehat{l(\varphi)} = \sum_{k=1}^{m}\int_{\Omega_k}\overline{u_k(x)}y_k(x)\,dx + c, \tag{1.58}$$

where $u_k\in L^2(\Omega_k),\ k=\overline{1,m},\ c\in\mathbb{C}$.

Denote by $u=(u_1,\ldots,u_m)$ an element belonging to $H := L^2(\Omega_1)\times\ldots\times L^2(\Omega_k)$.

Definition 1.1. *An estimate* $\widehat{\widehat{l(\varphi)}}$ *is called a minimax estimate of the* $l(\varphi)$ *if an element* $\hat{u}=(\hat{u}_1,\ldots,\hat{u}_m)\in H$ *and a number* $\hat{c}\in\mathbb{C}$ *are determined from the condition*

$$\sup_{(\tilde{f},\tilde{g})\in G_0,\,\tilde{\xi}\in G_1}\mathbb{E}[l(\tilde{\varphi})-\widehat{l(\tilde{\varphi})}]^2 \to \inf_{u\in H,\,c\in\mathbb{C}}.$$

Here $\tilde{\varphi}$ is a solution to problem (1.49)–(1.52) when $f = \tilde{f}$, $g = \tilde{g}$, and $\widehat{l(\tilde{\varphi})} = \sum_{k=1}^{m} \int_{\Omega_k} \overline{u_k(x)} \tilde{y}_k(x)\,dx + c$, $\tilde{y}_k(x) = \int_{\Omega_k} g_k(x,y)\tilde{\varphi}(y)\,dy + \tilde{\xi}_k(x)$, $x \in \Omega_k$, $k = \overline{1,m}$,

The quantity

$$\sigma := \{ \sup_{(\tilde{f},\tilde{g})\in G_0,\, \tilde{\xi}\in G_1} \mathbb{E}[l(\tilde{\varphi}) - \widehat{\widehat{l(\tilde{\varphi})}}]^2 \}^{1/2} \tag{1.59}$$

is called the error of the minimax estimation of $l(\varphi)$.

Thus, the minimax estimate is an estimate minimizing the maximal mean-square estimation error calculated for the "worst" implementation of perturbations.

1.3 Reduction of the estimation problem to an optimal control problem

Lemma 1.2. *The problem of finding the minimax estimate of $l(\varphi)$ is equivalent to the problem of optimal control of a system described by a BVP*

$$z(\cdot;u) \in H^1((\Omega_R \setminus \bar{\Omega}), \Delta), \tag{1.60}$$

$$-(\Delta + \bar{k}^2) z(x;u) = \chi_{\omega_0}(x) l_0(x) - \sum_{k=1}^{m} \chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(\eta,x)} u_k(\eta)\,d\eta \ \text{in}\ \Omega_R \setminus \bar{\Omega}, \tag{1.61}$$

$$\frac{\partial z(\cdot;u)}{\partial \nu} = 0 \ \text{on}\ \Gamma, \tag{1.62}$$

$$\frac{\partial z(\cdot;u)}{\partial r} = M_{\bar{k}}^{(2)} z(\cdot;u) \ \text{on}\ \Gamma_R \tag{1.63}$$

with the quality criterion

$$I(u) := \int_{\Omega_0} Q_1^{-1} z(x;u)\overline{z(x;u)}\,dx + \int_{\Gamma} Q_2^{-1} z(\cdot;u)\overline{z(\cdot;u)}\,d\Gamma$$

$$+ \sum_{k=1}^{m} \int_{\Omega_k} r_k^{-2}(x)|u_k(x)|^2\,dx \to \inf_{u\in H}, \tag{1.64}$$

where R is chosen so that $\bar{\Omega}_i \subset \Omega_R \setminus \bar{\Omega}$, $i = \overline{0,m}$, $\bar{\omega}_0 \subset \Omega_R \setminus \bar{\Omega}$.

Proof. Taking into account (1.56), (1.57), and (1.58), we obtain

$$l(\tilde{\varphi}) - \widehat{l(\tilde{\varphi})} = \int_{\omega_0} \overline{l_0(x)} \tilde{\varphi}(x)\,dx - \sum_{k=1}^{m} \int_{\Omega_k} \overline{u_k(x)} \tilde{y}_k(x)\,dx - c$$

$$= \int_{\omega_0} \overline{l_0(x)} \tilde{\varphi}(x)\,dx - \sum_{k=1}^{m} \int_{\Omega_k} \overline{u_k(x)} \int_{\Omega_k} g_k(x,y)\tilde{\varphi}(y)\,dy\,dx - \sum_{k=1}^{m} \int_{\Omega_k} \overline{u_k(x)} \tilde{\xi}_k(x)\,dx - c$$

$$= \int_{\omega_0} \overline{l_0(x)}\tilde{\varphi}(x)\,dx - \sum_{k=1}^m \int_{\Omega_k}\int_{\Omega_k} g_k(y,x)\overline{u_k(y)}\,dy\,\tilde{\varphi}(x)\,dx - \sum_{k=1}^m \int_{\Omega_k} \overline{u_k(x)}\tilde{\xi}_k(x)\,dx - c$$

$$= \int_{\Omega_R\setminus\bar{\Omega}} \overline{\left(\chi_{\omega_0}(x)l_0(x) - \sum_{k=1}^m \chi_{\Omega_k}(x)\int_{\Omega_k} \overline{g_k(y,x)}u_k(y)\,dy\right)}\tilde{\varphi}(x)\,dx$$

$$- \sum_{k=1}^m \int_{\Omega_k} \overline{u_k(x)}\tilde{\xi}_k(x)\,dx - c. \tag{1.65}$$

For any fixed $u = (u_1, \dots, u_m) \in H$ introduce the function $z(x;u)$ as a unique solution of problem (1.60)–(1.63). According to the equivalent variational formulation of this problem, it means that $z(x;u)$ satisfies the integral identity

$$a^*(z(\cdot;u),\theta) = \int_{\Omega_R\setminus\bar{\Omega}} (\nabla z(x;u)\nabla\overline{\theta}(x) - \bar{k}^2 z(x;u)\,\overline{\theta}(x))\,dx - \int_{\Gamma_R} M_{\bar{k}}^{(2)} z(\cdot;u)\,\overline{\theta}\,d\Gamma_R$$

$$= \int_{\Omega_R\setminus\bar{\Omega}} \left(\chi_{\omega_0}(x)l_0(x) - \sum_{k=1}^m \chi_{\Omega_k}(x)\int_{\Omega_k} \overline{g_k(y,x)}u_k(y)\,dy\right)\overline{\theta(x)}\,dx \quad \forall\theta \in H^1(\Omega_R \setminus \bar{\Omega}). \tag{1.66}$$

Set $\theta = \tilde{\varphi}$ in (1.66). Then we obtain

$$a^*(z(\cdot;u),\tilde{\varphi}) = \int_{\Omega_R\setminus\bar{\Omega}} (\nabla z(x;u)\nabla\overline{\tilde{\varphi}(x)} - \bar{k}^2 z(x;u)\,\overline{\tilde{\varphi}(x)}\,dx - \int_{\Gamma_R} M_{\bar{k}}^{(2)} z(\cdot;u)\,\overline{\tilde{\varphi}}\,d\Gamma_R$$

$$= \int_{\Omega_R\setminus\bar{\Omega}} \left(\chi_{\omega_0}(x)l_0(x) - \sum_{k=1}^m \chi_{\Omega_k}(x)\int_{\Omega_k} \overline{g_k(y,x)}u_k(y)\,dy\right)\overline{\tilde{\varphi}}\,dx. \tag{1.67}$$

On the other hand, since $\tilde{\varphi}$ is a solution of problem (1.49)–(1.52) with $f = \tilde{f}$ and $g = \tilde{g}$, setting $\psi = \varphi$ and $\theta = z(\cdot;u)$ in (1.19), we find

$$a(\tilde{\varphi}, z(\cdot;u)) = \int_{\Omega_R\setminus\bar{\Omega}} (\nabla\tilde{\varphi}(x)\nabla\overline{z(x;u)} - k^2\tilde{\varphi}(x)\,\overline{z(x;u)}\,dx - \int_{\Gamma_R} M_k^{(1)}\tilde{\varphi}\,\overline{z(\cdot;u)}\,d\Gamma_R$$

$$= \int_{\Omega_0} \tilde{f}(x)\overline{z(x;u)}\,dx + \int_\Gamma \tilde{g}\overline{z(\cdot;u)}\,d\Gamma. \tag{1.68}$$

By Lemma 1.1, $\overline{a^*(z,\tilde{\varphi})} = a(\tilde{\varphi}, z)$. This identity and (1.64), (1.67), and (1.68) imply

$$l(\tilde{\varphi}) - \widehat{l(\tilde{\varphi})} = \int_{\Omega_R\setminus\bar{\Omega}} \overline{\left(\chi_{\omega_0}(x)l_0(x) - \sum_{k=1}^m \chi_{\Omega_k}(x)\int_{\Omega_k} \overline{g_k(y,x)}u_k(y)\,dy\right)}\tilde{\varphi}(x)\,dx$$

$$= \int_{\Omega_0} \tilde{f}(x)\overline{z(x;u)}\,dx + \int_\Gamma \tilde{g}\overline{z(\cdot;u)}\,d\Gamma - \sum_{k=1}^m \int_{\Omega_k} \overline{u_k(x)}\tilde{\xi}_k(x)\,dx - c. \tag{1.69}$$

Taking into consideration the relationship $\mathbf{D}\eta=\mathbf{E}|\eta-\mathbf{E}\eta|^2=\mathbf{E}|\eta|^2-|\mathbf{E}\eta|^2$ that couples dispersion $\mathbf{D}\eta$ of the complex random variable $\eta=\eta_1+i\eta_2$ and its expectation $\mathbf{E}\eta=\mathbf{E}\eta_1+i\mathbf{E}\eta_2$, we obtain from the last formulas

$$\mathbf{E}\left|l(\tilde\varphi)-\widehat{l(\tilde\varphi)}\right|^2=\left|\int_{\Omega_0}\tilde f(x)\overline{z(x;u)}\,dx+\int_\Gamma \tilde g\overline{z(\cdot;u)}\,d\Gamma\right|^2+\mathbf{E}\left|\sum_{k=1}^m\int_{\Omega_k}\overline{u_k(x)}\,\tilde\xi_k(x)\,dx\right|^2.$$

Therefore,

$$\begin{aligned}\inf_{c\in\mathbb{C}}\sup_{(\tilde f,\tilde g)\in G_0,\tilde\xi\in G_1}&\mathbf{E}|l(\tilde\varphi)-\widehat{l(\tilde\varphi)}|^2=\\ =\inf_{c\in\mathbb{C}}\sup_{(\tilde f,\tilde g)\in G_0}&\left|\int_{\Omega_0}\tilde f(x)\overline{z(x;u)}\,dx+\int_\Gamma \tilde g\overline{z(\cdot;u)}\,d\Gamma-c\right|^2\\ &+\sup_{\tilde\xi\in G_1}\mathbf{E}\left|\sum_{k=1}^m\int_{\Omega_k}\overline{u_k(x)}\,\tilde\xi_k(x)\,dx\right|^2.\end{aligned}\tag{1.70}$$

In order to calculate the first term on the right-hand side of (1.70) make use of the generalized Cauchy$-$Bunyakovsky inequality in (1.53). We have

$$\inf_{c\in\mathbb{C}}\sup_{(\tilde f,\tilde g)\in G_0}\left|\int_{\Omega_0}\tilde f(x)\overline{z(x;u)}\,dx+\int_\Gamma \tilde g\overline{z(\cdot;u)}\,d\Gamma-c\right|^2$$

$$\begin{aligned}=\inf_{c\in\mathbb{C}}\sup_{(\tilde f,\tilde g)\in G_0}\Bigg|&\int_{\Omega_0}(\tilde f(x)-f_0(x))\overline{z(x;u)}\,dx+\int_\Gamma(\tilde g-g_0)\overline{z(\cdot;u)}\,d\Gamma\\ &+\int_{\Omega_0}f_0(x)\overline{z(x)}\,dx+\int_\Gamma g_0\overline{z(\cdot;u)}\,d\Gamma-c\Bigg|^2\end{aligned}$$

$$\begin{aligned}\le&\left\{\int_{\Omega_0}Q_1^{-1}z(x)\overline{z(x;u)}\,dx+\int_\Gamma Q_2^{-1}z(\cdot;u)\overline{z(\cdot;u)}\,d\Gamma\right\}\\ &\times\left\{\int_{\Omega_0}Q_1(\tilde f-f_0)(x)\overline{(\tilde f(x)-f_0(x))}\,dx+\int_\Gamma Q_2(\tilde g-g_0)\overline{(\tilde g-g_0)}\,d\Gamma\right\}\\ \le&\int_{\Omega_0}Q_1^{-1}z(x;u)\overline{z(x;u)}\,dx+\int_\Gamma Q_2^{-1}z(\cdot;u)\overline{z(\cdot;u)}\,d\Gamma.\end{aligned}\tag{1.71}$$

The direct substitution shows that that inequality (1.71) is transformed to an equality on the element $(\tilde f^{(0)}(\cdot),\tilde g^{(0)})$, where

$$\tilde f^{(0)}(x):=\frac{1}{d}Q_1^{-1}z(x;u)+f_0(x),$$

$$\tilde{g}^{(0)} := \frac{1}{d} Q_2^{-1} z(\cdot; u) + g_0,$$

$$d = \Big(\int_{\Omega_0} Q_1^{-1} z(x; u) \overline{z(x; u)}\, dx + \int_\Gamma Q_2^{-1} z(\cdot; u) \overline{z(\cdot; u)}\, d\Gamma \Big)^{1/2}.$$

Therefore

$$\inf_{c \in \mathbb{C}} \sup_{(\tilde{f}, \tilde{g}) \in G_0,\, \tilde{\xi} \in G_1} \left| \int_{\Omega_0} (\tilde{f}(x) - f_0(x)) \overline{z(x; u)}\, dx + \int_\Gamma (\tilde{g} - g_0) \overline{z(\cdot; u)}\, d\Gamma - c \right|^2$$

$$= \int_\Omega Q_1^{-1} z(x; u) \overline{z(x; u)}\, dx + \int_\Gamma Q_2^{-1} z(\cdot; u) \overline{z(\cdot; u)}\, d\Gamma \tag{1.72}$$

with

$$c = \int_{\Omega_0} \overline{z(x; u)} f_0(x)\, dx + \int_\Gamma \overline{z(\cdot; u)} g_0\, d\Gamma.$$

In order to calculate the second term on the right-hand side of (1.70), note that the Cauchy−Bunyakovsky inequality and (1.55) yield

$$\left| \sum_{k=1}^m \int_{\Omega_k} \overline{u_k(x)}\, \tilde{\xi}_k(x)\, dx \right|^2 \leq$$

$$\sum_{k=1}^m \int_{\Omega_k} r_k^{-2}(x) |u_k(x)|^2\, dx \sum_{k=1}^m \int_{\Omega_k} r_k^2(x) |\tilde{\xi}_k(x)|^2\, dx,$$

the latter implies

$$\sup_{\tilde{\xi} \in G_1} \mathbf{E} \left| \sum_{k=1}^m \int_{\Omega_k} \overline{u_k(x)}\, \tilde{\xi}_k(x)\, dx \right|^2 \leq \sum_{k=1}^m \int_{\Omega_k} r_k^{-2}(x) |u_k(x)|^2.$$

However

$$\mathbf{E} \left| \sum_{k=1}^m \int_{\Omega_k} \overline{u_k(x)}\, \tilde{\xi}_k^{(0)}(x)\, dx \right|^2 = \sum_{k=1}^m \int_{\Omega_k} r_k^{-2}(x) |u_k(x)|^2,$$

where

$$\tilde{\xi}_k^{(0)}(x) = \frac{\nu r_k^{-2}(x) u_k(x)}{\{\sum_{k=1}^m \int_{\Omega_k} r_k^{-2}(x) |u_k(x)\}^{1/2}},\ \ x \in \Omega_k,\ \ k = \overline{1, m},$$

$$\tilde{\xi}^{(0)} = (\tilde{\xi}_1^{(0)}(\cdot), \dots, \tilde{\xi}_m^{(0)}(\cdot)) \in G_1,$$

and ν is a random variable with $\mathbf{E}\nu = 0$ and $\mathbf{E}|\nu|^2 = 1$. Therefore

$$\sup_{\tilde{\xi} \in G_1} \mathbf{E} \left| \sum_{k=1}^m \int_{\Omega_k} \overline{u_k(x)}\, \tilde{\xi}_k(x)\, dx \right|^2 = \sum_{k=1}^m \int_{\Omega_k} r_k^{-2}(x) |u_k(x)|^2, \tag{1.73}$$

which proves the required assertion. The validity of Lemma 2.2 follows now from relationships (1.70), (1.72), and (1.73). $\square$

1.4 Representation of minimax estimates and estimation errors

In the course of the proof of Theorem 1.1 below, we show that the solution to the optimal control problem (1.2)–(1.64) (and therefore, the determination of the minimax estimate, in line with Theorem 1.2) is reduced to the solution of a certain integro-differential equation system. Namely, the following statement holds.

Theorem 1.1. *The minimax estimate of $l(\varphi)$ has the form*

$$\widehat{\widehat{l(\varphi)}} = \sum_{k=1}^{m} \int_{\Omega_k} \overline{\hat{u}_k(x)} y_k(x)\, dx + \hat{c}, \tag{1.74}$$

where

$$\hat{u}_k(x) = r_k^2(x) \int_{\Omega_k} g_k(x,y) p(y)\, dy,\ \ k = \overline{1,m},\ \ \hat{c} = \int_{\Omega_0} \overline{z(x)} f_0(x)\, dx + \int_{\Gamma} \overline{z} g_0\, d\Gamma, \tag{1.75}$$

and functions z and p are determined from the solution to the following problem:

$$z \in H^1((\Omega_R \setminus \bar{\Omega}), \Delta), \tag{1.76}$$

$$-(\Delta + \bar{k}^2) z(x) = \chi_{\omega_0}(x) l_0(x) - \sum_{k=1}^{m} \chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(\eta,x)} \hat{u}_k(\eta)\, d\eta \ \ in\ \Omega_R \setminus \bar{\Omega}, \tag{1.77}$$

$$\frac{\partial z}{\partial \nu} = 0 \ \ on\ \Gamma, \tag{1.78}$$

$$\frac{\partial z}{\partial r} = M_{\bar{k}}^{(2)} z \ \ on\ \Gamma_R, \tag{1.79}$$

$$p \in H^1((\Omega_R \setminus \bar{\Omega}), \Delta), \tag{1.80}$$

$$-(\Delta + k^2) p(x) = \chi_{\Omega_0}(x) Q_1^{-1} z(x) \ \ in\ \Omega_R \setminus \bar{\Omega}, \tag{1.81}$$

$$\frac{\partial p}{\partial \nu} = Q_2^{-1} z, \ \ on\ \Gamma, \tag{1.82}$$

$$\frac{\partial p}{\partial \nu} = M_k^{(1)} p \ \ on\ \Gamma_R. \tag{1.83}$$

Problem (1.76)–(1.83) is uniquely solvable. The restrictions of the solutions of this problem corresponding to $R = R_1$ and $R = R_2$ on $\Omega_{R=\min\{R_1,R_2\}}$ coincide with the solution corresponding to $R = \min\{R_1, R_2\}$.

The estimation error σ is determined by the formula[3]

$$\sigma = l(p)^{1/2} = \left(\int_{\omega_0} \overline{l_0(x)} p(x)\, dx \right)^{1/2}. \tag{1.84}$$

[3]In the proof of this theorem, we show that the value of $\int_{\omega_0} \overline{l_0(x)} p(x)\, dx$ is real.

Proof. Let us show first that the optimal control problem (1.60)–(1.64) is uniquely solvable; that is, there exists one and only one element $\hat{u} \in H$, at which functional (1.64) attains the minimum value, $I(\hat{u}) = \inf_{u \in H} I(u)$.

It is easy to see that the solution $z(x;u)$ of BVP (1.60)–(1.63) can be represented as

$$z(x;u) = z_0(x) + \tilde{z}(x;u), \tag{1.85}$$

where $z_0(x)$ and $\tilde{z}(x;u)$ are solutions of the following BVPs

$$z_0 \in H^1((\Omega_R \setminus \bar{\Omega}), \Delta), \tag{1.86}$$

$$-(\Delta + \bar{k}^2) z_0(x) = \chi_\omega(x) l_0(x) \text{ in } \Omega_R \setminus \bar{\Omega}, \tag{1.87}$$

$$\frac{\partial z_0}{\partial \nu} = 0 \text{ on } \Gamma, \tag{1.88}$$

$$\frac{\partial z_0}{\partial r} = M_{\bar{k}}^{(2)} z_0 \text{ on } \Gamma_R \tag{1.89}$$

and

$$\tilde{z}(\cdot;u) \in H^1((\Omega_R \setminus \bar{\Omega}), \Delta), \tag{1.90}$$

$$-(\Delta + \bar{k}^2) \tilde{z}(x;u) = -\sum_{k=1}^{m} \chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(\eta, x)} u_k(\eta)\, d\eta \text{ in } \Omega_R \setminus \bar{\Omega}, \tag{1.91}$$

$$\frac{\partial \tilde{z}(\cdot;u)}{\partial \nu} = 0 \text{ on } \Gamma, \tag{1.92}$$

$$\frac{\partial \tilde{z}(\cdot;u)}{\partial r} = M_{\bar{k}}^{(2)} \tilde{z}(\cdot;u) \text{ on } \Gamma_R. \tag{1.93}$$

Using representations (1.85) for $z(t;u)$, write functional $I(u)$ in the form

$$I(u) = \int_{\Omega_0} Q_1^{-1} z(x;u) \overline{z(x;u)}\, dx + \int_\Gamma Q_2^{-1} z(\cdot;u) \overline{z(\cdot;u)}\, d\Gamma + \sum_{k=1}^{m} \int_{\Omega_k} r_k^{-2}(x) |u_k(x)|^2\, dx$$

$$= \int_{\Omega_0} Q_1^{-1} (z_0(x) + \tilde{z}(x;u)) \overline{(z_0(x) + \tilde{z}(x;u))}\, dx + \int_\Gamma Q_2^{-1} (z_0 + \tilde{z}(\cdot;u)) \overline{(z_0 + \tilde{z}(\cdot;u))}\, d\Gamma$$

$$\sum_{k=1}^{m} \int_{\Omega_k} r_k^{-2}(x) |u_k(x)|^2\, dx = \tilde{I}(u) + L(u) + c_0,$$

where

$$\tilde{I}(u) := \int_{\Omega_0} Q_1^{-1} \tilde{z}(x;u) \overline{\tilde{z}(x;u)}\, dx + \int_\Gamma Q_2^{-1} \tilde{z}(\cdot;u) \overline{\tilde{z}(\cdot;u)}\, d\Gamma + \sum_{k=1}^{m} \int_{\Omega_k} r_k^{-2}(x) |u_k(x)|^2\, dx$$

is a quadratic functional in the space H which corresponds to a semi-bilinear continuous Hermitian form

$$\pi(u,v) = \int_{\Omega_0} Q_1^{-1}\tilde{z}(x;u)\overline{\tilde{z}(x;v)}\,dx + \int_{\Gamma} Q_2^{-1}\tilde{z}(\cdot;u)\overline{\tilde{z}(\cdot;v)}\,d\Gamma + \sum_{k=1}^{m}\int_{\Omega_k} r_k^{-2}(x)u_k(x)\overline{v_k(x)}\,dx \quad (1.94)$$

on $H \times H$ and satisfies

$$\tilde{I}(u) \geq a\|u\|_{H_0}^2 \;\forall u \in H,\; a = \text{const}; \quad (1.95)$$

note that

$$L(u) := 2\mathrm{Re}\int_{\Omega_0} Q_1^{-1}\tilde{z}(x;u)\overline{z_0(x)}\,dx + 2\mathrm{Re}\int_{\Gamma} Q_2^{-1}\tilde{z}(\cdot;u)\overline{z_0}\,d\Gamma$$

is a linear continuous functional in H and

$$c_0 := \int_{\Omega_0} Q_1^{-1}z_0(x)\overline{z_0(x)}\,dx + \int_{\Gamma} Q_2^{-1}z_0\overline{z_0}\,d\Gamma.$$

Prove, for example, the continuity of form (1.94); namely, the inequality

$$|\pi(v,w)| \leq c\|v\|_{H_0}\|w\|_{H_0} \quad \forall v,w \in V,\; c = \text{const} \quad (1.96)$$

(the continuity of linear functional $L(v)$ is proved in a similar manner).

Using estimate (1.34) obtained above and the Cauchy–Bunyakovsky inequality we have

$$|\pi(u,v)| \leq \left(\int_{\Omega_0} Q_1^{-1}\tilde{z}(x;u)\overline{\tilde{z}(x;u)}\,dx\right)^{1/2}\left(\int_{\Omega_0} Q_1^{-1}\tilde{z}(x;v)\overline{\tilde{z}(x;v)}\,dx\right)^{1/2}$$
$$+\left(\int_{\Gamma} Q_2^{-1}\tilde{z}(\cdot;u)\overline{\tilde{z}(\cdot;u)}\,d\Gamma\right)^{1/2}\left(\int_{\Gamma} Q_2^{-1}\tilde{z}(\cdot;v)\overline{\tilde{z}(\cdot;v)}\,d\Gamma\right)^{1/2}$$
$$+\left(\sum_{k=1}^{m}\int_{\Omega_k} r_k^{-2}(x)|u_k(x)|^2\,dx\right)^{1/2}\left(\sum_{k=1}^{m}\int_{\Omega_k} r_k^{-2}(x)|v_k(x)|^2\,dx\right)^{1/2}$$
$$\leq \left(\int_{\Omega_0} Q_1^{-1}\tilde{z}(x;u)\overline{\tilde{z}(x;u)}\,dx + \int_{\Gamma} Q_2^{-1}\tilde{z}(\cdot;u)\overline{\tilde{z}(\cdot;u)}\,d\Gamma + \sum_{k=1}^{m}\int_{\Omega_k} r_k^{-2}(x)|u_k(x)|^2\,dx\right)^{1/2}$$
$$\times\left(\int_{\Omega_0} Q_1^{-1}\tilde{z}(x;v)\overline{\tilde{z}(x;v)}\,dx + \int_{\Gamma} Q_2^{-1}\tilde{z}(\cdot;v)\overline{\tilde{z}(\cdot;v)}\,d\Gamma + \sum_{k=1}^{m}\int_{\Omega_k} r_k^{-2}(x)|v_k(x)|^2\,dx\right)^{1/2}$$

$$\le \left\{ \left(\int_{\Omega_0} |Q_1^{-1}\tilde{z}(x;u)|^2\,dx \right)^{1/2} \left(\int_{\Omega_0} |\tilde{z}(x;u)|^2\,dx \right)^{1/2} + \left(\int_\Gamma |Q_2^{-1}\tilde{z}(\cdot;u)|^2\,d\Gamma \right)^{1/2} \right.$$
$$\left. \times \left(\int_\Gamma |\tilde{z}(\cdot;u)|^2\,d\Gamma \right)^{1/2} + \left(\sum_{k=1}^m \int_{\Omega_k} r_k^{-4}(x)|u_k(x)|^2\,dx \right)^{1/2} \left(\sum_{k=1}^m \int_{\Omega_k} |u_k(x)|^2\,dx \right)^{1/2} \right\}^{1/2}$$

$$\times \left\{ \left(\int_{\Omega_0} |Q_1^{-1}\tilde{z}(x;v)|^2\,dx \right)^{1/2} \left(\int_{\Omega_0} |\tilde{z}(x;v)|^2\,dx \right)^{1/2} + \left(\int_\Gamma |Q_2^{-1}\tilde{z}(\cdot;v)|^2\,d\Gamma \right)^{1/2} \right.$$
$$\left. \times \left(\int_\Gamma |\tilde{z}(\cdot;v)|^2\,d\Gamma \right)^{1/2} + \left(\sum_{k=1}^m \int_{\Omega_k} r_k^{-4}(x)|v_k(x)|^2\,dx \right)^{1/2} \left(\sum_{k=1}^m \int_{\Omega_k} |v_k(x)|^2\,dx \right)^{1/2} \right\}^{1/2}$$

$$\le \max\{\|Q_1^{-1}\|, \|Q_2^{-1}\|, \beta\} \left\{ \int_{\Omega_0} |\tilde{z}(x;u)|^2\,dx + \int_\Gamma |\tilde{z}(\cdot;u)|^2\,d\Gamma + \sum_{k=1}^m \int_{\Omega_k} |u_k(x)|^2\,dx \right\}^{1/2}$$
$$\times \left\{ \int_{\Omega_0} |\tilde{z}(x;v)|^2\,dx + \int_\Gamma |\tilde{z}(\cdot;v)|^2\,d\Gamma + \sum_{k=1}^m \int_{\Omega_k} |v_k(x)|^2\,dx \right\}^{1/2}$$
$$\le \max\{\|Q_1^{-1}\|, \|Q_2^{-1}\|, \beta\} \left\{ \|\tilde{z}(\cdot;u)\|^2_{H^1(\Omega_0)} + \|\gamma_0\tilde{z}(\cdot;u)\|^2_{L^2(\Gamma)} + \|u\|^2_H \right\}^{1/2}$$
$$\times \left\{ \|\tilde{z}(\cdot;v)\|^2_{H^1(\Omega_0)} + \|\gamma_0\tilde{z}(\cdot;v)\|^2_{L^2(\Gamma)} + \|v\|^2_H \right\}^{1/2}, \tag{1.97}$$

where $\beta := \max_{1\le k\le m} \max_{x\in\bar{\Omega}_k} r_k^{-4}(x) > 0$.

Setting in (1.34) $\psi = \tilde{z}(\cdot;u)$, $f = -\sum_{k=1}^m \chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(\eta,x)} u_k(\eta)\,d\eta$, and $g = 0$, we find

$$\|\tilde{z}(\cdot;u)\|_{H^1(\Omega_0)} \le \|\tilde{z}(\cdot;u)\|_{H^1(\Omega_R\setminus\bar{\Omega})}$$
$$\le \alpha \left\| -\sum_{k=1}^m \chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(\eta,x)} u_k(\eta)\,d\eta \right\|_{L^2(\Omega_R\setminus\bar{\Omega})} \le c_1\|u\|_H, \tag{1.98}$$

where c_1 is a constant independent of R. The trace theorem and inequality (1.98) imply

$$\|\gamma_0\tilde{z}(\cdot;u)\|_{L^2(\Gamma)} \le c_2\|\tilde{z}(\cdot;u)\|_{H^1(\Omega_R\setminus\bar{\Omega})} \le c_3\|u\|_H, \quad c_2, c_3 = \text{const.} \tag{1.99}$$

From (1.97)–(1.98) we have

$$|\pi(u,v)| \le \max\{\|Q_1^{-1}\|, \|Q_2^{-1}\|, \beta\} (c_1\|u\|_H^2 + c_4\|u\|_H^2 + \|u\|_H^2)^{1/2}$$
$$\times (c_1\|v\|_H^2 + c_4\|v\|_H^2 + \|v\|_H^2)^{1/2} \le c\|u\|_H\|v\|_H,$$

where c is a constant independent of R.

Thus inequality (1.96) and, consequently, the continuity of form (1.94) are proved.

In line with Remark 1.4 to Theorem 1.1 proved in [1], p. 11, the latter statements imply the existence of the unique element $\hat{u} \in H$ such that

$$I(\hat{u}) = \inf_{u \in H} I(u).$$

Therefore, for any $\tau \in R$ and $v \in H$, the following relations are valid

$$\frac{d}{d\tau} I(\hat{u} + \tau v)\Big|_{\tau=0} = 0 \quad \text{and} \quad \frac{d}{d\tau} I(\hat{u} + i\tau v)\Big|_{\tau=0} = 0, \tag{1.100}$$

where $i = \sqrt{-1}$. Since $z(x; \hat{u} + \tau v) = z(x; \hat{u}) + \tau \tilde{z}(x; v)$, where $\tilde{z}(x; v)$ is the unique solution to BVP (1.90)–(1.93) at $u = v$ and $l_0 = 0$, the first relation in (1.100) yields

$$0 = \frac{1}{2}\frac{d}{d\tau} I(\hat{u} + \tau v)|_{\tau=0}$$

$$= \lim_{\tau \to 0} \frac{1}{2\tau} \Big\{ \Big[(Q_1^{-1} z(\cdot; \hat{u} + \tau v), z(\cdot; \hat{u} + \tau v))_{L^2(\Omega_0)} - (Q_1^{-1} z(\cdot; \hat{u}), z(\cdot; \hat{u}))_{L^2(\Omega_0)} \Big]$$
$$+ \Big[(Q_2^{-1} z(\cdot; \hat{u} + \tau v), z(\cdot; \hat{u} + \tau v))_{L^2(\Gamma)} - (Q_2^{-1} z(\cdot; \hat{u}), z(\cdot; \hat{u}))_{L^2(\Gamma)} \Big]$$
$$+ \Big[\sum_{k=1}^{m} \Big\{ \big(r_k^{-2}(\hat{u}_k + \tau v), (\hat{u}_k + \tau v)\big)_{L^2(\Omega_k)} - \big(r_k^{-2}\hat{u}_k, \hat{u}_k\big)_{L^2(\Omega_k)} \Big\} \Big] \Big\}$$

$$= \mathrm{Re} \Big\{ (Q_1^{-1} z(\cdot; \hat{u}), \tilde{z}(\cdot; v))_{L^2(\Omega_0)} + (Q_2^{-1} z(\cdot; \hat{u}), \tilde{z}(\cdot; v))_{\Gamma} + \sum_{i=1}^{m} \big(r_k^{-2} \hat{u}_k, v_k\big)_{L^2(\Omega_k)} \Big\}.$$

Similarly, taking into account that $z(x; \hat{u} + i\tau v) = z(x; \hat{u}) + i\tau \tilde{z}(x; v)$, we find

$$0 = \frac{1}{2}\frac{d}{d\tau} I(\hat{u} + i\tau v)|_{\tau=0}$$

$$= \mathrm{Im} \Big\{ (Q_1^{-1} z(\cdot; \hat{u}), \tilde{z}(\cdot; v))_{L^2(\Omega_0)} + (Q_2^{-1} z(\cdot; \hat{u}), \tilde{z}(\cdot; v))_{\Gamma} + \sum_{i=1}^{m} \big(r_k^{-2} \hat{u}_k, v_k\big)_{L^2(\Omega_k)} \Big\};$$

consequently,

$$(Q_1^{-1} z(\cdot; \hat{u}), \tilde{z}(\cdot; v))_{L^2(\Omega_0)} + (Q_2^{-1} z(\cdot; \hat{u}), \tilde{z}(\cdot; v))_{\Gamma} + \sum_{i=1}^{m} \big(r_k^{-2} \hat{u}_k, v_k\big)_{L^2(\Omega_k)} = 0. \tag{1.101}$$

Introduce a function $p(x) \in H^1((\Omega_R \setminus \bar{\Omega}), \Delta)$ as the unique solution to the BVP

$$-(\Delta + k^2) p(x) = \chi_{\Omega_0}(x) Q_1^{-1} z(x; \hat{u}) \text{ in } \Omega_R \setminus \bar{\Omega}, \tag{1.102}$$

$$\frac{\partial p}{\partial \nu} = Q_2^{-1} z(\cdot;\hat{u}) \text{ on } \Gamma, \tag{1.103}$$

$$\frac{\partial p}{\partial \nu} = M_k^{(1)} p \text{ on } \Gamma_R, \tag{1.104}$$

or to an equivalent variational problem

$$\begin{aligned} a(p,\theta) &= \int_{\Omega_R\setminus\bar{\Omega}} (\nabla p \nabla \overline{\theta} - k^2 p\, \overline{\theta})\, dx - \int_{\Gamma_R} M_k^{(1)} p\, \overline{\theta}\, d\Gamma_R \\ &= \int_{\Omega_R\setminus\bar{\Omega}} \chi_{\Omega_0}(x) Q_1^{-1} z(x;\hat{u}) \overline{\theta}\, dx + \int_{\Gamma_R} Q_2^{-1} z(\cdot;\hat{u}) \overline{\theta}\, d\Gamma_R \quad \forall \theta \in H^1(\Omega_R \setminus \bar{\Omega}). \end{aligned} \tag{1.105}$$

Setting in (1.105) $\theta = \tilde{z}(\cdot;v)$, we obtain

$$\begin{aligned} a(p,\tilde{z}(\cdot;v)) &= \int_{\Omega_R\setminus\bar{\Omega}} (\nabla p \nabla \overline{\tilde{z}(x;v)} - k^2 p\, \overline{\tilde{z}(x;v)})\, dx - \int_{\Gamma_R} M_k^{(1)} p\, \overline{\tilde{z}(\cdot;v)}\, d\Gamma_R \\ &= \int_{\Omega_R\setminus\bar{\Omega}} \chi_{\Omega_0}(x) Q_1^{-1} z(x;\hat{u}) \overline{\tilde{z}(x;v)}\, dx + \int_{\Gamma_R} Q_2^{-1} z(\cdot;\hat{u}) \overline{\tilde{z}(\cdot;v)}\, d\Gamma_R. \end{aligned} \tag{1.106}$$

Taking into account the fact that $\tilde{z}(\cdot;v)$ satisfies variation equation (1.27) with $\psi = \tilde{z}(\cdot;v)$ equivalent to BVP (1.90)–(1.93) with $u = v$ and putting in (1.27) $\theta = p$, we have

$$\begin{aligned} a^*(\tilde{z}(\cdot;v),p) &= \int_{\Omega_R\setminus\bar{\Omega}} (\nabla \tilde{z}(x;v) \nabla \overline{p}(x) - \bar{k}^2 \tilde{z}(x;v)\, \overline{p}(x))\, dx - \int_{\Gamma_R} M_{\bar{k}}^{(2)} \tilde{z}(\cdot;v)\, \overline{p}\, d\Gamma_R \\ &= -\int_{\Omega_R\setminus\bar{\Omega}} \sum_{k=1}^{m} \chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(y,x)} v_k(y)\, dy \overline{p(x)}\, dx. \end{aligned} \tag{1.107}$$

Since $a^*(\tilde{z}(\cdot;v),p) = \overline{a(p,\tilde{z}(\cdot;v))}$, we have

$$\begin{aligned} &\int_{\Omega_0} \overline{Q_1^{-1} z(x;\hat{u})} \tilde{z}(x;v)\, dx + \int_{\Gamma_R} \overline{Q_2^{-1} z(\cdot;\hat{u})} \tilde{z}(\cdot;v)\, d\Gamma_R \\ &= -\int_{\Omega_R\setminus\bar{\Omega}} \sum_{k=1}^{m} \chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(y,x)} v_k(y)\, dy \overline{p(x)}\, dx \\ &= -\sum_{k=1}^{m} \int_{\Omega_k} \left(\int_{\Omega_k} \overline{g_k(x,y) p(y)}\, dy \right) v_k(x)\, dx. \end{aligned} \tag{1.108}$$

From (1.101) it follows that

$$\overline{(Q_1^{-1} z(\cdot;\hat{u}), \tilde{z}(\cdot;v))_{L^2(\Omega_0)}} + \overline{(Q_2^{-1} z(\cdot;\hat{u}), \tilde{z}(\cdot;v))_{\Gamma}} = -\sum_{k=1}^{m} \int_{\Omega_k} r_k^{-2}(x) \overline{\hat{u}_k(x)} v_k(x)\, dx. \tag{1.109}$$

Relations (1.108) and (1.109) imply

$$\sum_{k=1}^{m}\int_{\Omega_k} r_k^{-2}(x)\overline{\hat{u}_k(x)}v_k(x)\,dx$$

$$=\sum_{k=1}^{m}\int_{\Omega_k}\left(\int_{\Omega_k}\overline{g_k(x,y)p(y)\,dy}\right)v_k(x)\,dx.\forall v_k\in L^2(\Omega_k),\;\; k=\overline{1,m}.$$

Hence,

$$\hat{u}_k(x)=r_k^2(x)\int_{\Omega_k} g_k(x,y)p(y)\,dy.$$

Now let us establish the validity of formula (1.84). We have

$$\sigma^2 := \sup_{(\tilde{f},\tilde{g})\in G_0,\,\tilde{\xi}\in G_1} M[l(\varphi)-\widehat{\widehat{l(\varphi)}}]^2 =$$

$$\int_{\Omega_0} Q_1^{-1}z(x)\overline{z(x)}\,dx+\int_{\Gamma} Q_2^{-1}z\overline{z}\,d\Gamma+\sum_{k=1}^{m}\int_{\Omega_k}\hat{r}_k^2(x)\left|\int_{\Omega_k} g_k(x,y)p(y)\,dy\right|^2 dx$$

Transform the sum of first two terms. Make use of equalities (1.132)–(1.135) to obtain

$$a(p,z)=\int_{\Omega_R\setminus\bar{\Omega}}\chi_{\Omega_0}(x)Q_1^{-1}z(x)\overline{z(x)}\,dx+\int_{\Gamma} Q_2^{-1}z\overline{z}\,d\Gamma, \tag{1.110}$$

hence

$$\sigma^2=a(p,z)+\sum_{k=1}^{m}\int_{\Omega_k}\hat{r}_k^2(x)\left|\int_{\Omega_k} g_k(x,y)p(y)\,dy\right|^2 dx. \tag{1.111}$$

Note that z satisfies (1.76)–(1.79) which yields an integral identity

$$a^*(z,\theta)=\int_{\Omega_R\setminus\bar{\Omega}}\Big(\chi_{\omega_0}(x)l_0(x)$$

$$-\sum_{k=1}^{m}\chi_{\Omega_k}(x)\int_{\Omega_k}\overline{g_k(y,x)}\hat{u}_k(y)\,dy\Big)\overline{\theta(x)}\,dx\quad \forall\theta\in H^1(\Omega_R\setminus\bar{\Omega}); \tag{1.112}$$

setting $\theta=p$, we find

$$a^*(z,p)=\int_{\Omega_R\setminus\bar{\Omega}}\left(\chi_{\omega_0}(x)l_0(x)-\sum_{k=1}^{m}\chi_{\Omega_k}(x)\int_{\Omega_k}\overline{g_k(y,x)}\hat{u}_k(y)\,dy\right)\overline{p(x)}\,dx.$$

From the latter relations, the formula $\overline{a^*(z,p)}=a(p,z)$, and (1.111) it follows that

$$\sigma^2=\int_{\omega_0}\overline{l_0(x)}p(x)\,dx-\sum_{k=1}^{m}\int_{\Omega_k}\int_{\Omega_k} g_k(y,x)\overline{\hat{u}_k(y)}\,dyp(x)\,dx$$

$$+\sum_{k=1}^{m}\int_{\Omega_k}\hat{r}_k^2(x)\left|\int_{\Omega_k}g_k(x,y)p(y)\,dy\right|^2\,dx.$$

However,

$$\hat{u}_k(y)=r_k^2(y)\int_{\Omega_k}g_k(y,\eta)p(\eta)\,d\eta,$$

therefore,

$$\sum_{k=1}^{m}\int_{\Omega_k}\int_{\Omega_k}g_k(y,x)\overline{\hat{u}_k(y)}\,dyp(x)\,dx$$

$$=\sum_{k=1}^{m}\int_{\Omega_k}\int_{\Omega_k}g_k(y,x)r_k^2(y)\overline{\int_{\Omega_k}g_k(y,\eta)p(\eta)\,d\eta}dyp(x)\,dx$$

$$=\sum_{k=1}^{m}\int_{\Omega_k}r_k^2(x)\int_{\Omega_k}g_k(x,y)p(y)\,dy\overline{\int_{\Omega_k}g_k(x,\eta)p(\eta)\,d\eta}\,dx$$

$$=\sum_{k=1}^{m}\int_{\Omega_k}\hat{r}_k^2(x)\left|\int_{\Omega_k}g_k(x,y)p(y)\,dy\right|^2\,dx.$$

Finally, we obtain

$$\sigma^2=\int_{\omega_0}\overline{l_0(x)}p(x)\,dx.$$

□

In the following theorem we obtain an alternative representation for minimax estimate $\widehat{\widehat{l(\varphi)}}$ that does not depend on the form of functional l.

Theorem 1.2. *The minimax estimate of $l(\varphi)$ has the form*

$$\widehat{\widehat{l(\varphi)}}=l(\hat{\varphi}(\cdot,\omega)), \tag{1.113}$$

where function $\hat{\varphi}$ is a solution to the following problem:

$$\hat{p}\in L^2(\Sigma,H^1((\Omega_R\setminus\bar{\Omega}),\Delta)), \tag{1.114}$$

$$\begin{aligned}&-(\Delta+\bar{k}^2)\hat{p}(x,\omega)\\&\quad=\chi_{\Omega_k}(x)\sum_{k=1}^{m}\int_{\Omega_k}r_k^2(\tau)\overline{g_k(\tau,x)}\left[y_k(\tau,\omega)-\int_{\Omega_k}\hat{\varphi}(\eta,\omega)g_k(\tau,\eta)d\eta\right]d\tau\ \textit{in}\ \Omega_R\setminus\bar{\Omega},\end{aligned} \tag{1.115}$$

$$\frac{\partial\hat{p}(\cdot,\omega)}{\partial\nu}=0\ \textit{on}\ \Gamma, \tag{1.116}$$

$$\frac{\partial \hat{p}(\cdot,\omega)}{\partial r} = M_{\bar{k}}^{(2)}\hat{p}(\cdot,\omega) \ on \ \Gamma_R \tag{1.117}$$

$$\hat{\varphi} \in L^2(\Sigma, H^1((\Omega_R \setminus \bar{\Omega}), \Delta)), \tag{1.118}$$

$$-(\Delta + k^2)\hat{\varphi}(x,\omega) = \chi_{\Omega_0}(x)Q_1^{-1}\hat{p}(x,\omega) + f_0(x) \ in \ \Omega_R \setminus \bar{\Omega}, \tag{1.119}$$

$$\frac{\partial \hat{\varphi}(\cdot,\omega)}{\partial \nu} = Q_2^{-1}\hat{p}(\cdot,\omega) + g_0, \ on \ \Gamma, \tag{1.120}$$

$$\frac{\partial \hat{\varphi}(\cdot,\omega)}{\partial \nu} = M_k^{(1)}\hat{\varphi}(\cdot,\omega) \ on \ \Gamma_R, \tag{1.121}$$

where equalities (1.115)−(1.117) and (1.119)−(1.121) are fulfilled with probability 1. *Problem* (1.114)−(1.121) *is uniquely solvable. The restrictions of the solutions of this problem on* $\Omega_{R=\min\{R_1,R_2\}}$ *corresponding to* $R = R_1$ *and* $R = R_2$ *coincide with the solution corresponding to* $R = \min\{R_1, R_2\}$.

Proof. The proof is similar to the that of Theorem 2.1. Consider the problem of optimal control of the equation system

$$\hat{p}(\cdot,\cdot;u) \in L^2(\Sigma, H^1((\Omega_R \setminus \bar{\Omega}), \Delta)), \tag{1.122}$$

$$-(\Delta + \bar{k}^2)\hat{p}(x,\omega;u) = d_0(x,\omega) - \sum_{k=1}^{m}\int_{\Omega_k} \chi_{\Omega_k}(x)\overline{g_k(\tau,x)}u(\tau,\omega)d\tau \text{ in } \Omega_R \setminus \bar{\Omega}, \tag{1.123}$$

$$\frac{\partial \hat{p}(\cdot,\omega;u)}{\partial \nu} = 0 \text{ on } \Gamma, \tag{1.124}$$

$$\frac{\partial \hat{p}(\cdot,\omega;u)}{\partial r} = M_{\bar{k}}^{(2)}\hat{p}(\cdot,\omega;u) \text{ on } \Gamma_R \tag{1.125}$$

with the cost function

$$I(u) = \mathbb{E}\Big\{\int_{\Omega_0} Q_1^{-1}(\hat{p}(\cdot,\omega;u) + Q_1 f_0)(x)\overline{(\hat{p}(\cdot,\omega;u) + Q_1 f_0)(x)}\,dx$$
$$+\int_{\Gamma} Q_2^{-1}(\hat{p}(\cdot,\omega;u) + Q_2 g_0)(x)\overline{(\hat{p}(\cdot,\omega;u) + Q_2 g_0)(x)}\,d\Gamma$$
$$+\sum_{k=1}^{m}\int_{\Omega_k} r_k^{-2}(x)|u_k(x;\omega)|^2\,dx\Big\} \to \inf_{u\in L^2(\Sigma,H)}. \tag{1.126}$$

where

$$d_0(x,\omega) := \sum_{k=1}^{m}\int_{\Omega_k} \chi_{\Omega_k}(x)\overline{g_k(\tau,x)}r_k^2(\tau)y_k(\tau;\varphi,\xi_k(\omega))\,d\tau,$$

The form of functional $I(u)$ and proof of Theorem 2.1 suggest that there is one and only one element $\hat{u} \in L^2(\Sigma, H)$ such that

$$I(\hat{u}) = \inf_{u \in L^2(\Sigma,H)} I(u).$$

Next, denoting by $\hat{\varphi}(t;\omega)$ the unique solution to the BVP

$$\hat{\varphi}(\cdot,\omega) \in L^2(\Sigma, X_{\Omega_R\setminus\bar{\Omega}}),$$

$$-(\Delta + k^2)\hat{\varphi}(x,\omega) = \chi_{\Omega_0}(x)Q_1^{-1}\hat{p}(x,\omega;\hat{u}) + f_0(x) \text{ in } \Omega_R \setminus \bar{\Omega},$$

$$\frac{\partial\hat{\varphi}(\cdot,\omega)}{\partial\nu} = Q_2^{-1}\hat{p}(\cdot,\omega;\hat{u}) + g_0, \text{ on } \Gamma,$$

$$\frac{\partial\hat{\varphi}(\cdot,\omega)}{\partial\nu} = M_k^{(1)}\hat{\varphi}(\cdot,\omega) \text{ on } \Gamma_R,$$

and making use of virtually the same reasoning that led to the proof of Theorem 2.1 (by applying estimate (1.40) instead of (1.34)), we arrive at the equality $\hat{u}(\tau,\omega) = \int_{\Omega_k}\hat{\varphi}(\eta,\omega)g_k(\tau,\eta)d\eta$. Denoting $\hat{p}(x,\omega) = \hat{p}(x,\omega;\hat{u})$, we deduce from the latter statement the unique solvability of BVP (1.114)–(1.121).

Now let us prove the representation $\widehat{\widehat{l(\varphi)}} = l(\hat{\varphi})$. By virtue of (1.56) and (1.75),

$$\widehat{\widehat{l(\varphi)}} = \sum_{k=1}^{m}\int_{\Omega_k}\overline{\hat{u}_k(x)}y_k(x;\varphi,\xi_k(\omega)) + \hat{c}$$

$$= \sum_{k=1}^{m}\int_{\Omega_k}\int_{\Omega_k} r_k^2(x)\overline{g_k(x,\eta)p(\eta)}\,d\eta\, y_k(x;\varphi,\xi_k(\omega))dx + \hat{c}$$

$$= \sum_{k=1}^{m}\int_{\Omega_k}\int_{\Omega_k} r_k^2(\tau)\overline{g_k(\tau,x)}y_k(\tau;\varphi,\xi_k(\omega))\overline{p(x)}\,d\tau\,dx + \hat{c}. \tag{1.127}$$

The function $\hat{p}(x,\omega;\hat{u}) := \hat{p}(x,\omega)$ is a solution to BVP (1.122)–(1.125) with $u = \hat{u}$, therefore $\forall\theta \in H^1(\Omega_R \setminus \Omega)$ the following identity holds

$$a^*(\hat{p}(\cdot,\omega),\theta(\cdot) = \int_{\Omega_R\setminus\bar{\Omega}}(\nabla\hat{p}(x,\omega)\nabla\overline{\theta(x)}) - \bar{k}^2\hat{p}(x,\omega)\,\overline{\theta(x)})\,dx$$
$$- \int_{\Gamma_R} M_{\bar{k}}^{(2)}\hat{p}(\cdot,\omega)\,\overline{\theta(\cdot)}\,d\Gamma_R$$

$$= \int_{\Omega_R\setminus\Omega}\sum_{k=1}^{m}\int_{\Omega_k}\chi_{\Omega_k}(x)r_k^2(\tau)\overline{g_k(\tau,x)}\,[y_k(\tau,\omega,\xi_k(\omega))$$
$$- \int_{\Omega_k}\hat{\varphi}(\eta,\omega)g_k(\tau,\eta)d\eta\Big]\,d\tau\overline{\theta(x}\,dx. \tag{1.128}$$

Setting in this identity $\theta(x) = p(x)$ we obtain

$$a^*(\hat{p}(\cdot,\omega), p(\cdot)) = \int_{\Omega_R\setminus\bar{\Omega}} (\nabla \hat{p}(x,\omega)\nabla\overline{p(x)}) - \bar{k}^2\hat{p}(x,\omega)\,\overline{p(x)})\,dx \\ - \int_{\Gamma_R} M_{\bar{k}}^{(2)}\hat{p}(\cdot,\omega)\,\overline{p(\cdot)}\,d\Gamma_R$$

$$= \int_{\Omega_R\setminus\Omega}\sum_{k=1}^{m}\int_{\Omega_k}\chi_{\Omega_k}(x)r_k^2(\tau)\overline{g_k(\tau,x)}\,(y_k(\tau,\varphi,\xi_k(\omega)) \\ - \int_{\Omega_k}\hat{\varphi}(\eta,\omega)g_k(\tau,\eta)d\eta\Big)\,d\tau\overline{p(x)}\,dx. \quad (1.129)$$

Since $p(x)$ satisfies (1.80)$-$(1.83) and consequently (1.105), we have

$$a(p,\theta) = \int_{\Omega_R\setminus\bar{\Omega}} (\nabla p(x)\nabla\overline{\theta(x)} - k^2 p(x)\,\overline{\theta(x)})\,dx - \int_{\Gamma_R} M_k^{(1)} p\,\overline{\theta}\,d\Gamma_R \\ = \int_{\Omega_R\setminus\bar{\Omega}} \chi_{\Omega_0}(x)Q_1^{-1}z(x)\overline{\theta(x)}\,dx + \int_{\Gamma} Q_2^{-1}z\overline{\theta(\cdot)}\,d\Gamma \quad \forall\theta\in H^1(\Omega_R\setminus\bar{\Omega}),$$

which yields

$$a(p,\hat{p}(\cdot,\omega) = \int_{\Omega_R\setminus\bar{\Omega}} (\nabla p(x)\nabla\overline{\hat{p}(x,\omega)} - k^2 p(x)\,\overline{\hat{p}(x,\omega)})\,dx - \int_{\Gamma_R} M_k^{(1)} p\,\overline{\hat{p}(\cdot,\omega)}\,d\Gamma_R \\ = \int_{\Omega_R\setminus\bar{\Omega}} \chi_{\Omega_0}(x)Q_1^{-1}z(x)\overline{\hat{p}(x,\omega)}\,dx + \int_{\Gamma} Q_2^{-1}z\overline{\hat{p}(\cdot,\omega)}\,d\Gamma. \quad (1.130)$$

Since $a^*(\hat{p}(\cdot;\omega),p) = \overline{a(p,\hat{p}(\cdot;\omega))}$, (1.129) and (1.130) imply

$$\sum_{k=1}^{m}\int_{\Omega_k}\int_{\Omega_k} r_k^2(\tau)\overline{g_k(\tau,x)}\left(y_k(\tau;\varphi,\xi_k(\omega)) - \int_{\Omega_k}\hat{\varphi}(\eta,\omega)g_k(\tau,\eta)\,d\eta\right)d\tau\overline{p(x)}\,dx \\ = \int_{\Omega_R\setminus\bar{\Omega}} \chi_{\Omega_0}(x)Q_1^{-1}\hat{p}(x,\omega)\overline{z(x)}\,dx + \int_{\Gamma} Q_2^{-1}\hat{p}(\cdot,\omega)\overline{z}\,d\Gamma. \quad (1.131)$$

Equating (1.127) and (1.131), we obtain

$$\widehat{\overline{l(\varphi)}} - \hat{c} - \sum_{k=1}^{m}\int_{\Omega_k}\int_{\Omega_k} r_k^2(\tau)\overline{g_k(\tau,x)}\int_{\Omega_k}\hat{\varphi}(\eta,\omega)g_k(\tau,\eta)\,d\eta\,d\tau\overline{p(x)}\,dx \\ = \int_{\Omega_R\setminus\bar{\Omega}} \chi_{\Omega_0}(x)Q_1^{-1}\overline{z(x)}\hat{p}(x,\omega)\,dx + \int_{\Gamma} Q_2^{-1}\overline{z}\hat{p}(\cdot,\omega)\,d\Gamma. \quad (1.132)$$

Next, since $\hat\varphi(\cdot,\omega)$ and z satisfy, respectively, equalities (1.118)–(1.121) and (1.76)–(1.79), these functions satisfy also the identities

$$a(\hat\varphi(\cdot,\omega),\theta)=\int_{\Omega_R\setminus\bar\Omega}\chi_{\Omega_0}(x)\left(Q_1^{-1}\hat p(x,\omega)+f_0(x)\right)\overline{\theta(x)}\,dx + \int_\Gamma\left(Q_2^{-1}\hat p(\cdot,\omega)+g_0\right)\overline{\theta}\,d\Gamma \quad (1.133)$$

and

$$a^*(z,\theta)=\int_{\Omega_R\setminus\bar\Omega}\Bigg(\chi_{\omega_0}(x)l_0(x) - \sum_{k=1}^m\chi_{\Omega_k}(x)\int_{\Omega_k}r_k^2(\eta)\overline{g_k(\eta,x)}\int_{\Omega_k}g_k(\eta,\varsigma)p(\varsigma)\,d\varsigma\,d\eta\Bigg)\overline{\theta(x)}\,dx. \quad (1.134)$$

Setting in (1.133) $\theta(x)=z(x)$ and in (1.134) $\theta(x)=\hat\varphi(x,\omega)$ and taking into notice that $\overline{a^*(z,\hat\varphi(\cdot,\omega))}=a(\hat\varphi(\cdot,\omega),z)$, we have

$$\int_{\Omega_R\setminus\bar\Omega}\chi_{\Omega_0}(x)Q_1^{-1}\hat p(x,\omega)\overline{z(x)}dx+\int_\Gamma Q_2^{-1}\hat p(\cdot,\omega)\overline{z}\,d\Gamma+\int_{\Omega_0}f_0(x)\overline{z(x)}\,dx+\int_\Gamma g_0\overline{z}\,d\Gamma$$

$$=\int_{\Omega_R\setminus\bar\Omega}\chi_{\omega_0}(x)l_0(x)\hat\varphi(x,\omega))\,dx - \int_{\Omega_R\setminus\bar\Omega}\sum_{k=1}^m\chi_{\Omega_k}(\eta)\int_{\Omega_k}r_k^2(\tau)\overline{g_k(\tau,x)}\int_{\Omega_k}g_k(\tau,\eta)\hat\varphi(\eta,\omega))\,d\eta\,d\tau\overline{p(x)}\,dx. \quad (1.135)$$

Representation (1.113) follows now from (1.132) and (1.135) if we take into account (1.75). □

Remark 1. *If we define a minimax estimate $\hat\varphi(x,\omega)$ of the unknown solution $\varphi(x)$ of BVP (1.49)–(1.52) as the estimate linear with respect to observations (1.56), which is determined from the condition of minimum of the maximal mean square error of the estimate taken over sets G_0 and G_1, then it may be shown that, under certain restrictions on G_0 and G_1, this minimax estimate of $\varphi(x)$ coincides with the function $\hat\varphi(x,\omega)$ obtained from the solution to problem* (1.114)–(1.121).

1.5 Minimax estimation of the right-hand sides of equalities that enter the statement of the boundary value problem. Representations for minimax estimates and estimation errors

The problem is to determine a minimax estimate of the value of the functional

$$l(F) = \int_{\Omega_0} \overline{l_0(x)} f(x)\, dx + \int_\Gamma \overline{l_1} g\, d\Gamma \tag{1.136}$$

from observations (1.56) in the class of estimates linear with respect to observations

$$\widehat{l(F)} = \sum_{k=1}^m \int_{\Omega_k} \overline{u_k(x)} y_k(x)\, dx + c, \tag{1.137}$$

where $u_k \in L^2(\Omega_k)$, $k = \overline{1,m}$, $c \in \mathbb{C}$, and $l_0 \in L^2(\Omega_0)$ and $l_1 \in L^2(\Gamma)$ are given functions, under the assumption that $F := (f, g) \in G_0$ and the errors $\xi(\cdot) = (\xi_1(\cdot), \dots, \xi_m(\cdot))$ in observations (1.56) belong to G_1, where sets G_0 and G_1 are defined by (1.53), (1.54), and (1.55), respectively.

Definition 1.2. *The estimate of the form*

$$\widehat{\widehat{l(F)}} = \sum_{k=1}^m \int_{\Omega_k} \overline{\hat{u}_k(x)} y_k(x)\, dx + \hat{c}, \tag{1.138}$$

will be called the minimax estimate of $l(F)$ if the element $\hat{u} = (u_1, \dots, u_m) \in H$ and number $\hat{c} \in \mathbb{C}$ are determined form the condition

$$\sup_{\tilde{F} \in G_0, \tilde{\eta} \in G_1} \mathbf{E}|l(\tilde{F}) - \widehat{l(\tilde{F})}|^2 \to \inf_{u \in H,\, c \in \mathbb{C}}.$$

Here

$$\widehat{l(\tilde{F})} = \sum_{k=1}^m \int_{\Omega_k} \overline{u_k(x)} \tilde{y}_k(x)\, dx + c, \tag{1.139}$$

$\tilde{y}_k(x) = \int_{\Omega_k} g_k(x,y) \tilde{\varphi}(y)\, dy + \tilde{\xi}_k(x)$, $x \in \Omega_k$, $k = \overline{1,m}$, *and* $\tilde{\varphi}$ *is a solution to BVP* (1.49)–(1.52) *at* $f = \tilde{f}$ *and* $g = \tilde{g}$. *The quantity*

$$\sigma := \{ \sup_{(\tilde{f},\tilde{g}) \in G_0,\, \tilde{\xi} \in G_1} \mathbb{E}[l(\tilde{F}) - \widehat{\widehat{l(\tilde{F})}}]^2 \}^{1/2}$$

will be called the error of the minimax estimation of expression (1.136).

Lemma 1.3. *Finding the minimax estimate of $l(F)$ is equivalent to the problem of optimal control of a system described by the BVP*

$$z(\cdot;u) \in H^1((\Omega_R \setminus \bar{\Omega}), \Delta), \tag{1.140}$$

$$\Delta z(x;u) + \bar{k}^2 z(x;u) = \sum_{k=1}^{m} \chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(\eta, x)} u_k(\eta)\, d\eta \text{ in } \Omega_R \setminus \bar{\Omega}, \tag{1.141}$$

$$\frac{\partial z(\cdot;u)}{\partial \nu} = 0 \text{ on } \Gamma, \tag{1.142}$$

$$\frac{\partial z(\cdot;u)}{\partial r} = M_{\bar{k}}^{(2)} z(\cdot;u) \text{ on } \Gamma_R \tag{1.143}$$

with the quality criterion

$$I(u) := \int_{\Omega_0} Q_1^{-1}(l_0 + z(\cdot;u))(x)\overline{(l_0(x) + z(x;u)))}\, dx$$
$$+ \int_{\Gamma} (Q_2^{-1}(l_1 + z(\cdot;u)))\overline{(l_1 + z(\cdot;u))}\, d\Gamma + \sum_{k=1}^{m} \int_{\Omega_k} r_k^{-2}(x)|u_k(x)|^2\, dx \to \inf_{u \in H}. \tag{1.144}$$

Proof. Taking into account (1.136), (1.137) and (1.56), we obtain

$$l(\tilde{F}) - \widehat{l(\tilde{F})} = \int_{\Omega_0} \overline{l_0(x)} \tilde{f}(x)\, dx + \int_{\Gamma} \overline{l_1} g\, d\Gamma - \sum_{k=1}^{m} \int_{\Omega_k} \overline{u_k(x)} y_k(x; \tilde{\varphi}, \tilde{\xi}_k)\, dx - c$$

$$= \int_{\Omega_0} \overline{l_0(x)} \tilde{f}\, dx + \int_{\Gamma} \overline{l_1} g\, d\Gamma - \sum_{k=1}^{m} \int_{\Omega_k} \overline{u_k(x)} \int_{\Omega_k} g_k(x,y) \tilde{\varphi}(y)\, dy\, dx$$
$$- \sum_{k=1}^{m} \int_{\Omega_k} \overline{u_k(x)} \tilde{\xi}_k(x)\, dx - c$$

$$= \int_{\Omega_0} \overline{l_0(x)} \tilde{f}(x)\, dx + \int_{\Gamma} \overline{l_1} g\, d\Gamma - \sum_{k=1}^{m} \int_{\Omega_k} \int_{\Omega_k} g_k(y,x) \overline{u_k(y)}\, dy\, \tilde{\varphi}(x)\, dx$$
$$- \sum_{k=1}^{m} \int_{\Omega_k} \overline{u_k(x)} \tilde{\xi}_k(x)\, dx - c$$

$$= \int_{\Omega_0} \overline{l_0(x)} f(x)\, dx + \int_{\Gamma} \overline{l_1} g\, d\Gamma - \sum_{k=1}^{m} \int_{\Omega_R \setminus \bar{\Omega}} \overline{\left(\chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(y,x)} u_k(y)\, dy \right)} \tilde{\varphi}(x)\, dx$$
$$- \sum_{k=1}^{m} \int_{\Omega_k} \overline{u_k(x)} \tilde{\xi}_k(x)\, dx - c. \tag{1.145}$$

For any fixed $u = (u_1, \dots, u_m) \in H$ introduce the function $z(x;u)$ as a unique solution of problem (1.140)–(1.143). According to the equivalent variational formulation of this problem it means that $z(x;u)$ satisfies the integral identity

$$a^*(z(\cdot;u),\theta) = \int_{\Omega_R\setminus\bar\Omega} (\nabla z(x;u)\nabla\overline{\theta}(x) - \bar{k}^2 z(x;u)\,\overline{\theta}(x))\,dx - \int_{\Gamma_R} M_{\bar{k}}^{(2)} z(\cdot;u)\,\overline{\theta}\,d\Gamma_R$$

$$= -\int_{\Omega_R\setminus\bar\Omega} \sum_{k=1}^m \chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(y,x)}u_k(y)\,dy\overline{\theta(x)}\,dx \quad \forall\theta \in H^1(\Omega_R\setminus\bar\Omega). \tag{1.146}$$

Set $\theta = \tilde\varphi$ in (1.66) to obtain

$$a^*(z(\cdot;u),\tilde\varphi) = \int_{\Omega_R\setminus\bar\Omega} (\nabla z(x;u)\nabla\overline{\tilde\varphi(x)} - \bar{k}^2 z(x;u)\,\overline{\tilde\varphi(x)}\,dx - \int_{\Gamma_R} M_{\bar{k}}^{(2)} z(\cdot;u)\,\overline{\tilde\varphi}\,d\Gamma_R$$

$$= -\int_{\Omega_R\setminus\bar\Omega} \sum_{k=1}^m \chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(y,x)}u_k(y)\,dy\overline{\tilde\varphi}\,dx. \tag{1.147}$$

On the other hand, since $\tilde\varphi$ is a solution of problem (1.49)–(1.52) with $f = \tilde f$ and $g = \tilde g$, setting $\psi = \varphi$ and $\theta = z(\cdot;u)$ in (1.19), we find

$$a(\tilde\varphi, z(\cdot;u)) = \int_{\Omega_R\setminus\bar\Omega} (\nabla\tilde\varphi(x)\nabla\overline{z(x;u)} - k^2\tilde\varphi(x)\,\overline{z(x;u)}\,dx - \int_{\Gamma_R} M_k^{(1)}\tilde\varphi\,\overline{z(\cdot;u)}\,d\Gamma_R$$

$$= \int_{\Omega_0} \tilde f(x)\overline{z(x;u)}\,dx + \int_\Gamma \tilde g\overline{z(\cdot;u)}\,d\Gamma. \tag{1.148}$$

By Lemma 1, $\overline{a^*(z,\tilde\varphi)} = a(\tilde\varphi, z)$. This identity together with (1.145), (1.147), and (1.148) imply

$$l(\tilde F) - \widehat{l(\tilde F)} = \int_{\Omega_0} \overline{l_0(x)}\tilde f(x)\,dx + \int_\Gamma \overline{l_1}\tilde g\,d\Gamma$$

$$-\int_{\Omega_R\setminus\bar\Omega} \overline{\sum_{k=1}^m \chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(y,x)}u_k(y)\,dy}\tilde\varphi(x)\,dx - \sum_{k=1}^m \int_{\Omega_k} \overline{u_k(x)}\tilde\xi_k(x)\,dx - c$$

$$= \int_{\Omega_0} \overline{l_0(x)}\tilde f(x)\,dx + \int_\Gamma \overline{l_1}\tilde g\,d\Gamma$$

$$+\int_{\Omega_0} \tilde f(x)\overline{z(x;u)}\,dx + \int_\Gamma \tilde g\overline{z(\cdot;u)}\,d\Gamma - \sum_{k=1}^m \int_{\Omega_k} \overline{u_k(x)}\tilde\xi_k(x)\,dx - c.$$

$$= \int_{\Omega_0} \tilde f(x)\overline{(l_0(x) + z(x;u))}\,dx + \int_\Gamma \tilde g\overline{(l_1 + z(\cdot;u))}\,d\Gamma - \sum_{k=1}^m \int_{\Omega_k} \overline{u_k(x)}\tilde\xi_k(x)\,dx - c.$$

The latter yields

$$\mathbf{E}\left|l(\tilde F)-\widehat{l(\tilde F)}\right|^2=\left|\int_{\Omega_0}\tilde f(x)\overline{(l_0(x)+z(x;u))}\,dx+\int_\Gamma \tilde g\overline{(l_1+z(\cdot;u))}\,d\Gamma\right|^2$$

$$+\mathbf{E}\left|\sum_{k=1}^m\int_{\Omega_k}\overline{u_k(x)}\,\tilde\xi_k(x)\,dx\right|^2.$$

Therefore,

$$\inf_{c\in\mathbb{C}}\sup_{(\tilde f,\tilde g)\in G_0,\tilde\xi\in G_1}\mathbf{E}|l(\tilde F)-\widehat{l(\tilde F)}|^2=$$

$$=\inf_{c\in\mathbb{C}}\sup_{(\tilde f,\tilde g)\in G_0}\left|\int_{\Omega_0}\tilde f(x)\overline{(l_0(x)+z(x;u))}\,dx+\int_\Gamma \tilde g\overline{(l_1+z(\cdot;u))}\,d\Gamma-c\right|^2$$

$$+\sup_{\tilde\xi\in G_1}\mathbf{E}\left|\sum_{k=1}^m\int_{\Omega_k}\overline{u_k(x)}\,\tilde\xi_k(x)\,dx\right|^2.$$

Beginning from this place, we apply the same reasoning as in the proof of Lemma 1.2 (replacing $z(x,u)$ by $l_0(x)+z(x,u)$) to obtain

$$\inf_{c\in\mathbb{C}}\sup_{(\tilde f,\tilde g)\in G_0,\tilde\xi\in G_1}\mathbf{E}|l(\tilde F)-\widehat{l(\tilde F)}|^2=I(u),$$

where $I(u)$ is determined by formula (1.144) for

$$c=\int_{\Omega_0}f_0(x)\overline{(l_0(x)+z(x;u))}\,dx+\int_\Gamma g_0\overline{(l_1+z(\cdot;u))}\,d\Gamma.$$

□

The following result follows from this lemma,

Theorem 1.3. *The minimax estimate of $l(F)$ has the form*

$$\widehat{\widehat{l(F)}}=\sum_{k=1}^m\int_{\Omega_k}\overline{\hat u_k(x)}y_k(x)\,dx+\hat c, \tag{1.149}$$

where

$$\hat u_k(x)=r_k^2(x)\int_{\Omega_k}g_k(x,y)p(y)\,dy,\ k=\overline{1,m}, \tag{1.150}$$

$$\hat c=\int_{\Omega_0}\overline{(z(x)+l_0(x))}f_0(x)\,dx+\int_\Gamma\overline{(z+l_1)}g_0\,d\Gamma, \tag{1.151}$$

and functions z and p are determined from the solution to the following problem:

$$z\in H^1((\Omega_R\setminus\bar\Omega),\Delta), \tag{1.152}$$

$$\Delta z(x) + \bar{k}^2 z(x) = \sum_{k=1}^{m} \chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(\eta, x)} \hat{u}_k(\eta)\, d\eta \ in\ \Omega_R \setminus \bar{\Omega}, \tag{1.153}$$

$$\frac{\partial z}{\partial \nu} = 0 \ on\ \Gamma, \tag{1.154}$$

$$\frac{\partial z}{\partial r} = M_{\bar{k}}^{(2)} z \ on\ \Gamma_R, \tag{1.155}$$

$$p \in H^1((\Omega_R \setminus \bar{\Omega}), \Delta), \tag{1.156}$$

$$-(\Delta + k^2) p = \chi_{\Omega_0} Q_1^{-1}(z + l_0) \ in\ \Omega_R \setminus \bar{\Omega}, \tag{1.157}$$

$$\frac{\partial p}{\partial \nu} = Q_2^{-1}(z + l_1), \ \ on\ \Gamma, \tag{1.158}$$

$$\frac{\partial p}{\partial \nu} = M_k^{(1)} p \ on\ \Gamma_R. \tag{1.159}$$

Problem (1.152)–(1.159) *is uniquely solvable. The restrictions of the solutions of this problem on* $\Omega_{R=\min\{R_1,R_2\}}$ *corresponding to* $R = R_1$ *and* $R = R_2$ *coincide with the solution corresponding to* $R = \min\{R_1, R_2\}$.

Estimation error σ *is determined by the formula* $\sigma = l(P)^{1/2}$, *where* $P = (Q_1^{-1}(l_0 + z), Q_2^{-1}(l_1 + \gamma_D z))$.

Proof. Similarly to the proof of Theorem 2.1, we will show that the solution to the optimal control problem (1.140)–(1.144) can be reduced to the solution of the equation system (1.152)–(1.159). To this end, note that there exists the unique element $\hat{u} \in H$ at which functional (1.144) attains its minimum, namely, $I(\hat{u}) = \inf_{u \in H} I(u)$. In order to prove this statement, represent $I(u)$ as

$$I(u) = \tilde{I}(u) + L(u) + \tilde{c}, \tag{1.160}$$

where

$$\tilde{I}(u) = \int_{\Omega_0} Q_1^{-1} z(x; u) \overline{z(x; u)}\, dx + \int_{\Gamma} Q_2^{-1} z(\cdot; u) \overline{z(\cdot; u)}\, d\Gamma + \sum_{k=1}^{m} \int_{\Omega_k} r_k^{-2}(x) |u_k(x)|^2\, dx$$

is a quadratic functional in space H corresponding to a semi-bilinear continuous Hermitian form [4]

$$\pi(u, v) = \int_{\Omega_0} Q_1^{-1} \tilde{z}(x; u) \overline{\tilde{z}(x; v)}\, dx + \int_{\Gamma} Q_2^{-1} \tilde{z}(\cdot; u) \overline{\tilde{z}(\cdot; v)}\, d\Gamma$$

[4]Its continuity is shown in the course of the proof of Theorem 1.1.

$$+\sum_{k=1}^{m}\int_{\Omega_k} r_k^{-2}(x)u_k(x)\overline{v_k(x)}\,dx \tag{1.161}$$

on $H\times H$ which satisfies the inequality

$$\tilde{I}(u)\geq a\|u\|^2_{H_0}\ \forall u\in H,\ a=\text{const}, \tag{1.162}$$

$$L(u):=2\text{Re}\int_{\Omega_0} Q_1^{-1}\tilde{z}(x;u)\overline{l_0(x)}\,dx+2\text{Re}\int_{\Gamma} Q_2^{-1}\tilde{z}(\cdot;u)\overline{l_1}\,d\Gamma,$$

is a linear continuous functional in H, and

$$\tilde{c}=\int_{\Omega_0} Q_1^{-1}l_0(x)\overline{l_0(x)}\,dx+\int_{\Gamma} Q_2^{-1}l_1\overline{l_1}\,d\Gamma.$$

This statement yields (see page 24) the existence of the unique element $\hat{u}\in H$ such that

$$I(\hat{u})=\inf_{u\in H} I(u).$$

Therefore, for any $\tau\in R$ and $v\in H$ the relations

$$\frac{d}{d\tau}I(\hat{u}+\tau v)\Big|_{\tau=0}=0\quad\text{and}\quad \frac{d}{d\tau}I(\hat{u}+i\tau v)\Big|_{\tau=0}=0 \tag{1.163}$$

hold. Taking into account that functions $z(x;\hat{u}+\tau v)$ and $z(x;\hat{u}+i\tau v)$ can be written, respectively, as $z(x;\hat{u}+\tau v)=z(x;\hat{u})+\tau\tilde{z}(x;v)$ and $z(x;\hat{u}+i\tau v)=z(x;\hat{u})+i\tau\tilde{z}(x;v)$, where $\tilde{z}(x;v)$ is the unique solution to problem (1.90)–(1.93) at $u=v$, we deduce from (1.163) that

$$(Q_1^{-1}(l_0+z(\cdot;\hat{u})),\tilde{z}(\cdot;v))_{L^2(\Omega_0)}+(Q_2^{-1}(l_1+z(\cdot;\hat{u})),\tilde{z}(\cdot;v))_\Gamma$$
$$+\sum_{k=1}^{m}\left(r_k^{-2}\hat{u}_k,v_k\right)_{L^2(\Omega_k)}=0. \tag{1.164}$$

Introduce the function $p(x)\in H^1((\Omega_R\setminus\bar{\Omega}),\Delta)$ as the unique solution to the BVP

$$-(\Delta+k^2)p(x)=\chi_{\Omega_0}(x)Q_1^{-1}(z(\cdot;\hat{u})+l_0)(x)\ \text{in}\ \Omega_R\setminus\bar{\Omega}, \tag{1.165}$$

$$\frac{\partial p}{\partial\nu}=Q_2^{-1}(z(\cdot;\hat{u})+l_1)\ \text{on}\ \Gamma, \tag{1.166}$$

$$\frac{\partial p}{\partial\nu}=M_k^{(1)}p\ \text{on}\ \Gamma_R, \tag{1.167}$$

or to the equivalent variational problem

$$a(p,\theta) = \int_{\Omega_R\setminus\bar\Omega} (\nabla p \nabla\overline{\theta} - k^2 p\,\overline{\theta})\,dx - \int_{\Gamma_R} M_k^{(1)} p\,\overline{\theta}\,d\Gamma_R$$
$$= \int_{\Omega_R\setminus\bar\Omega} \chi_{\Omega_0}(x) Q_1^{-1}(z(\cdot;\hat u)+l_0)(x)\overline{\theta}\,dx + \int_{\Gamma_R} Q_2^{-1}(z(\cdot;\hat u)+l_1)\overline{\theta}\,d\Gamma_R \quad \forall\theta\in H^1(\Omega_R\setminus\bar\Omega). \tag{1.168}$$

Setting in (1.168) $\theta = \tilde z(\cdot;v)$, we obtain

$$a(p,\tilde z(\cdot;v)) = \int_{\Omega_R\setminus\bar\Omega} (\nabla p\nabla\overline{\tilde z(x;v)} - k^2 p\,\overline{\tilde z(x;v)})\,dx - \int_{\Gamma_R} M_k^{(1)} p\,\overline{\tilde z(\cdot;v)}\,d\Gamma_R$$
$$= \int_{\Omega_R\setminus\bar\Omega} \chi_{\Omega_0}(x) Q_1^{-1}(z(\cdot;\hat u)+l_0)(x)\overline{\tilde z(x;v)}\,dx + \int_{\Gamma_R} Q_2^{-1}(z(\cdot;\hat u)+l_1)\overline{\tilde z(\cdot;v)}\,d\Gamma_R. \tag{1.169}$$

Taking into account the fact that $\tilde z(\cdot;v)$ satisfies variation equation (1.27) with $\psi = \tilde z(\cdot:v)$ and $f(x) = -\sum_{k=1}^m \chi_{\Omega_k}(x)\int_{\Omega_k}\overline{g_k(y,x)}v_k(y)$ which is equivalent to BVP (1.140)–(1.143) with $u=v$ and setting in (1.27) $\theta = p$, we have

$$a^*(\tilde z(\cdot;v),p) = \int_{\Omega_R\setminus\bar\Omega} (\nabla\tilde z(x;v)\nabla\overline{p}(x) - \bar k^2\tilde z(x;v)\,\overline{p}(x))\,dx - \int_{\Gamma_R} M_{\bar k}^{(2)}\tilde z(\cdot;v)\,\overline{p}\,d\Gamma_R$$
$$= -\int_{\Omega_R\setminus\bar\Omega}\sum_{k=1}^m \chi_{\Omega_k}(x)\int_{\Omega_k}\overline{g_k(y,x)}v_k(y)\,dy\overline{p(x)}\,dx. \tag{1.170}$$

Since $a^*(\tilde z(\cdot;v),p) = \overline{a(p,\tilde z(\cdot;v))}$, we obtain

$$\int_{\Omega_0}\overline{Q_1^{-1}(l_0+z(\cdot;\hat u))(x)}\tilde z(x;v)\,dx + \int_{\Gamma_R}\overline{Q_2^{-1}(l_1+z(\cdot;\hat u))}\tilde z(\cdot;v)\,d\Gamma_R$$
$$= -\int_{\Omega_R\setminus\bar\Omega}\sum_{k=1}^m \chi_{\Omega_k}(x)\int_{\Omega_k}\overline{g_k(y,x)}v_k(y)\,dy\overline{p(x)}\,dx$$
$$= -\sum_{k=1}^m\int_{\Omega_k}\left(\int_{\Omega_k}\overline{g_k(x,y)p(y)}\,dy\right)v_k(x)\,dx. \tag{1.171}$$

From (1.164) it follows that

$$\overline{(Q_1^{-1}(l_0+z(\cdot;\hat u)), l_0+\tilde z(\cdot;v))_{L^2(\Omega_0)}} + \overline{(Q_2^{-1}(l_1+z(\cdot;\hat u)), \tilde z(\cdot;v))_\Gamma}$$
$$= -\sum_{k=1}^m\int_{\Omega_k} r_k^{-2}(x)\overline{\hat u_k(x)}v_k(x)\,dx. \tag{1.172}$$

Relations (1.171) and (1.172) imply

$$\sum_{k=1}^{m} \int_{\Omega_k} r_k^{-2}(x)\overline{\hat{u}_k(x)} v_k(x)\,dx$$

$$= \sum_{k=1}^{m} \int_{\Omega_k} \left(\int_{\Omega_k} \overline{g_k(x,y)p(y)\,dy} \right) v_k(x)\,dx. \forall v_k \in L^2(\Omega_k),\;\; k = \overline{1,m}.$$

Hence,

$$\hat{u}_k(x) = r_k^2(x) \int_{\Omega_k} g_k(x,y)p(y)\,dy.$$

Now we can determine estimation error σ. Substituting

$$\hat{u}_k(x) = r_k^2(x) \int_{\Omega_k} g_k(x,y)p(y)\,dy,\;\; k = \overline{1,m},$$

to the formula $I(\hat{u}) = \sigma^2$, in which $I(u)$ is calculated according to (1.144) we obtain, taking into notice that $z(t) = z(t;\hat{u})$,

$$\sigma^2 = \int_{\Omega_0} Q_1^{-1}(l_0+z)(x)\overline{(l_0(x)+z(x))}\,dx + \int_{\Gamma} Q_2^{-1}(l_1+z)\overline{(l_1+z)}\,d\Gamma$$

$$+ \sum_{k=1}^{m} \int_{\Omega_k} \hat{r}_k^2(x) \left| \int_{\Omega_k} g_k(x,y)p(y)\,dy \right|^2 dx.$$

Transform the sum of the first two terms. To do this, make use of equalities (1.156)−(1.159) yielding

$$a(p,z) = \int_{\Omega_R\setminus\bar{\Omega}} \chi_{\Omega_0}(x)Q_1^{-1}(l_0+z)(x)\overline{z(x)}\,dx + \int_{\Gamma} Q_2^{-1}(l_1+z)\bar{z}\,d\Gamma, \tag{1.173}$$

so that

$$\sigma^2 = a(p,z) + \int_{\Omega_R\setminus\bar{\Omega}} \chi_{\Omega_0}(x)Q_1^{-1}(l_0+z)(x)\overline{l_0(x)}\,dx + \int_{\Gamma} Q_2^{-1}(l_1+z)\overline{l_1}\,d\Gamma$$

$$+ \sum_{k=1}^{m} \int_{\Omega_k} \hat{r}_k^2(x) \left| \int_{\Omega_k} g_k(x,y)p(y)\,dy \right|^2 dx. \tag{1.174}$$

However, because z satisfies (1.152)−(1.155), the following integral identity holds

$$a^*(z,\theta) = -\int_{\Omega_R\setminus\bar{\Omega}} \sum_{k=1}^{m} \chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(y,x)}\hat{u}_k(y)\,dy\overline{\theta(x)}\,dx \quad \forall\theta \in H^1(\Omega_R\setminus\bar{\Omega});$$

setting in this identity $\theta = p$, we have

$$a^*(z,p) = -\int_{\Omega_R\setminus\bar{\Omega}} \sum_{k=1}^{m} \chi_{\Omega_k}(x) \int_{\Omega_k} \overline{g_k(y,x)}\hat{u}_k(y)\,dy\overline{p(x)}\,dx.$$

The last formula, the relation $\overline{a^*(z,p)} = a(p,z)$, and (1.174) give us

$$\sigma^2 = \int_{\Omega_0} \overline{l_0(x)} Q_1^{-1}(l_0+z)(x)\,dx \int_\Gamma \overline{l_1} Q_2^{-1}(l_1+z)\,d\Gamma$$

$$-\sum_{k=1}^m \int_{\Omega_k}\int_{\Omega_k} g_k(y,x)\overline{\hat{u}_k(y)}\,dy p(x)\,dx + \sum_{k=1}^m \int_{\Omega_k} \hat{r}_k^2(x) \left| \int_{\Omega_k} g_k(x,y)p(y)\,dy \right|^2 dx.$$

Repeating literally the end of the proof of Theorem 2.3 we obtain

$$\sum_{k=1}^m \int_{\Omega_k}\int_{\Omega_k} g_k(y,x)\overline{\hat{u}_k(y)}\,dy p(x)\,dx$$

$$= \sum_{k=1}^m \int_{\Omega_k} \hat{r}_k^2(x) \left| \int_{\Omega_k} g_k(x,y)p(y)\,dy \right|^2 dx,$$

and finally

$$\sigma^2 = \int_{\Omega_0} \overline{l_0(x)} Q_1^{-1}(l_0+z)(x)\,dx \int_\Gamma \overline{l_1} Q_2^{-1}(l_1+z)\,d\Gamma = l(P).$$

□

In Theorem 1.4 stated below we obtain another representation for minimax estimate $\widehat{\widehat{l(F)}}$, not depending on the form of functional l.

Theorem 1.4. *The minimax estimate of $l(F)$ has the form*

$$\widehat{\widehat{l(F)}} = l(\hat{F}), \tag{1.175}$$

where $\hat{F} = (\hat{f}, \hat{g})$, $\hat{f}(x) = \hat{f}(x,\omega) = Q_1^{-1}\hat{p}(x,\omega) + f_0(x)$, $\hat{g} = \hat{g}(\cdot,\omega) = Q_2^{-1}\gamma_D\hat{p}(\cdot,\omega) + g_0$, and function $\hat{p} = \hat{p}(\cdot,\omega)$ is determined from the solution to problem (1.114)$-$(1.121)*.*

Proof. Taking into notice (1.149)$-$(1.151), we find

$$\widehat{\widehat{l(F)}} = \sum_{k=1}^m \int_{\Omega_k} \overline{\hat{u}_k(x)} y_k(x)\,dx + \hat{c}$$

$$= \sum_{k=1}^m \int_{\Omega_k}\int_{\Omega_k} r_k^2(\tau)\overline{g_k(\tau,x)} y_k(\tau)\overline{p(x)}\,d\tau\,dx + \hat{c}, \tag{1.176}$$

where $\hat{c}$ is determined from (1.151).

Next, repeating literally the proof of Theorem 1.2 on page 30, we arrive at the relatioship

$$a^*(\hat{p}(\cdot,\omega),p(\cdot)) = \int_{\Omega_R\setminus\bar{\Omega}} (\nabla\hat{p}(x,\omega)\nabla\overline{p(x)}) - \bar{k}^2\hat{p}(x,\omega)\,\overline{p(x)})\,dx \\ - \int_{\Gamma_R} M^{(2)}_{\bar{k}}\hat{p}(\cdot,\omega)\,\overline{p(\cdot)}\,d\Gamma_R$$

$$= \int_{\Omega_R\setminus\Omega}\sum_{k=1}^{m}\int_{\Omega_k}\chi_{\Omega_k}(x)r_k^2(\tau)\overline{g_k(\tau,x)}\Big(y_k(\tau) \\ - \int_{\Omega_k}\hat{\varphi}(\eta,\omega)g_k(\tau,\eta)d\eta\Big)d\tau\overline{p(x)}\,dx. \quad (1.177)$$

Taking into account that $p(x)$ satisfies (1.156)−(1.159), we obtain

$$a(p,\theta) = \int_{\Omega_R\setminus\bar{\Omega}}(\nabla p(x)\nabla\overline{\theta(x)} - k^2p(x)\,\overline{\theta(x)})\,dx - \int_{\Gamma_R} M^{(1)}_k p\,\overline{\theta}\,d\Gamma_R \\ = \int_{\Omega_R\setminus\bar{\Omega}}\chi_{\Omega_0}(x)Q_1^{-1}(z+l_0)(x)\overline{\theta(x)}\,dx + \int_\Gamma Q_2^{-1}(z+l_1)\overline{\theta(\cdot)}\,d\Gamma \quad \forall\theta\in H^1(\Omega_R\setminus\bar{\Omega});$$

consequently,

$$a(p,\hat{p}(\cdot,\omega) = \int_{\Omega_R\setminus\bar{\Omega}}(\nabla p(x)\nabla\overline{\hat{p}(x,\omega)} - k^2p(x)\,\overline{\hat{p}(x,\omega)})\,dx - \int_{\Gamma_R} M^{(1)}_k p\,\overline{\hat{p}(\cdot,\omega)}\,d\Gamma_R \\ = \int_{\Omega_R\setminus\bar{\Omega}}\chi_{\Omega_0}(x)Q_1^{-1}(z+l_0)(x)\overline{\hat{p}(x,\omega)}\,dx + \int_\Gamma Q_2^{-1}(z+l_1)\overline{\hat{p}(\cdot,\omega)}\,d\Gamma. \quad (1.178)$$

From (1.177) and the latter equality, it follows that

$$\sum_{k=1}^{m}\int_{\Omega_k}\int_{\Omega_k} r_k^2(\tau)\overline{g_k(\tau,x)}\left(y_k(\tau) - \int_{\Omega_k}\hat{\varphi}(\eta,\omega)g_k(\tau,\eta)\,d\eta\right)d\tau\overline{p(x)}\,dx \\ = \int_{\Omega_R\setminus\bar{\Omega}}\chi_{\Omega_0}(x)Q_1^{-1}\hat{p}(x,\omega)\overline{(z(x)+l_0(x))}\,dx + \int_\Gamma Q_2^{-1}\hat{p}(\cdot,\omega)\overline{(z+l_1)}\,d\Gamma. \quad (1.179)$$

Equating (1.176) and (1.179), we find

$$\widehat{\widehat{l(F)}} - \hat{c} - \sum_{k=1}^{m}\int_{\Omega_k}\int_{\Omega_k} r_k^2(\tau)\overline{g_k(\tau,x)}\int_{\Omega_k}\hat{\varphi}(\eta,\omega)g_k(\tau,\eta)\,d\eta\,d\tau\overline{p(x)}\,dx \\ + \int_{\Omega_R\setminus\bar{\Omega}}\chi_{\Omega_0}(x)Q_1^{-1}\overline{l_0}(x)\hat{p}(x,\omega)\,dx + \int_\Gamma Q_2^{-1}\overline{l_1}\hat{p}(\cdot,\omega)\,d\Gamma$$

$$= \int_{\Omega_R\setminus\bar\Omega} \chi_{\Omega_0}(x) Q_1^{-1}\overline{z}(x)\hat p(x,\omega)\,dx + \int_\Gamma Q_2^{-1}\overline{z}\hat p(\cdot,\omega)\,d\Gamma. \tag{1.180}$$

Next, since $\hat\varphi(\cdot,\omega)$ and z satisfy, respectively, equalities (1.118)–(1.121) and (1.152)–(1.155) these functions also satisfy the identities

$$a(\hat\varphi(\cdot,\omega),\theta) = \int_{\Omega_R\setminus\bar\Omega} \chi_{\Omega_0}(x)\left(Q_1^{-1}\hat p(x,\omega) + f_0(x)\right)\overline{\theta(x)}\,dx + \int_\Gamma \left(Q_2^{-1}\hat p(\cdot,\omega) + g_0\right)\overline{\theta}\,d\Gamma \tag{1.181}$$

and

$$a^*(z,\theta) = -\int_{\Omega_R\setminus\bar\Omega} \sum_{k=1}^m \chi_{\Omega_k}(x)\int_{\Omega_k} r_k^2(\eta)\overline{g_k(\eta,x)}\int_{\Omega_k} g_k(\eta,\varsigma)p(\varsigma)\,d\varsigma\,d\eta\overline{\theta(x)}\,dx. \tag{1.182}$$

Setting in (1.181) $\theta(x) = z(x)$ and in (1.182) $\theta(x) = \hat\varphi(x,\omega)$ and taking into account that $\overline{a^*(z,\hat\varphi(\cdot,\omega))} = a(\hat\varphi(\cdot,\omega),z)$, we obtain

$$\int_{\Omega_R\setminus\bar\Omega} \chi_{\Omega_0}(x)Q_1^{-1}\hat p(x,\omega)\overline{z(x)}dx + \int_\Gamma Q_2^{-1}\hat p(\cdot,\omega)\overline{z}\,d\Gamma + \int_{\Omega_0} f_0(x)\overline{z(x)}\,dx + \int_\Gamma g_0\overline{z}\,d\Gamma$$

$$= -\int_{\Omega_R\setminus\bar\Omega} \sum_{k=1}^m \chi_{\Omega_k}(\eta)\int_{\Omega_k} r_k^2(\tau)\overline{g_k(\tau,x)}\int_{\Omega_k} g_k(\tau,\eta)\hat\varphi(\eta,\omega))\,d\eta\,d\tau\overline{p(x)}\,dx. \tag{1.183}$$

From (1.180) and (1.183), it follows, in view of (1.151), that

$$\widehat{\widehat{l(F)}} - \int_{\Omega_0}\overline{z(x)}f_0(x)\,dx - \int_\Gamma \overline{z}g_0\,d\Gamma - \int_{\Omega_0}\overline{l_0(x)}f_0(x)\,dx - \int_\Gamma \overline{l_1}g_0\,d\Gamma$$

$$+ \int_{\Omega_0}\overline{z(x)}f_0(x)\,dx + \int_\Gamma \overline{z}g_0\,d\Gamma$$

$$= \int_{\Omega_0}\overline{l_0(x)}Q_1^{-1}\hat p(x,\omega)\,dx + \int_\Gamma \overline{l_1}Q_2^{-1}\hat p(\cdot,\omega)\,d\Gamma,$$

thus

$$\widehat{\widehat{l(F)}} = \int_{\Omega_0}\overline{l_0(x)}(f_0(x) + Q_1^{-1}\hat p(x,\omega))\,dx + \int_\Gamma \overline{l_1}(l_1 + Q_2^{-1}\hat p(\cdot,\omega))\,d\Gamma$$

$$= \int_{\Omega_0}\overline{l_0(x)}\hat f(x)\,dx + \int_\Gamma \overline{l_1}\hat g\,d\Gamma = l(\hat F).$$

□

Remark 2. *If we define a minimax estimate $\hat F(x,\omega)$ of the element*[5] $F = (f,g)$ *as an estimate linear with respect to observations (1.56), which is determined from*

[5]Here f and g are the functions entering the statement of BVP (1.49)–(1.52) and $F = (f,g) \in G_0$.

the condition of minimum of the maximal mean square error of the estimate taken over sets G_0 and G_1, then it may be established that, under certain restrictions on G_0 and G_1, this minimax estimate of F coincides with the element $\hat{F} = (\hat{f}, \hat{g})$, where $\hat{f} = Q_1^{-1}\hat{p}(x,\omega) + f_0(x)$ and $\hat{g} = Q_2^{-1}\gamma_D\hat{p}(\cdot,\omega) + g_0$, and the function $\hat{p} = \hat{p}(\cdot,\omega)$ is determined from the solution to problem (1.114)–(1.121).

Using Theorems 1.1–1.4 together with the solution techniques employing the so-called DtN finite-element methods elaborated for problems (1.14)–(1.16) and (1.41)–(1.43), one can construct algorithms of numerical solution to problems (1.76)–(1.83), (1.114)–(1.121) and (1.152)–(1.159) and obtain the required minimax estimates.

Remark 3. *All results of this section remain valid in the three-dimensional case. For example, finding minimax estimates of the solutions to the BVPs for the Helmholtz equation that describe diffraction of acoustic waves by an obstacle $\Omega \in \mathbb{R}^3$ can be reduced to the solution of integro-differential equation systems* (1.76)–(1.83) *and* (1.76)–(1.83)*; the domain $\mathbb{R}^2 \setminus \bar{\Omega}$, should be replaced by $\mathbb{R}^3 \setminus \bar{\Omega}$, and plane domains Ω_i, $i = \overline{1,k}$, on which observations are made should be considered, as well as supports Ω_0 and ω_0 of functions f and l_0, as spatial domains. The Sommerfeld condition*

$$\frac{\partial\varphi}{\partial r} - ik\varphi = o(1/r^{1/2}),\ r = |x| = \sqrt{x_1^2 + x_2^2},\ r \to \infty \tag{1.184}$$

should be replaced by

$$\frac{\partial\varphi}{\partial r} - ik\varphi = o(1/r),\ r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2},\ r \to \infty; \tag{1.185}$$

operators $M_k^{(j)}\psi$ defined on a circle Γ_R should be replaced by the following operators defined on a sphere Γ_R of radius R

$$(M_k^{(j)}\psi)(R,\theta,\phi) := \frac{k}{4\pi}\sum_{n=0}^{\infty}\frac{h_n^{(j)\prime}(kR)}{h_n^{(j)}(kR)}\sum_{n=-m}^{m} u_{mn}Y_{mn}(\theta,\phi), \tag{1.186}$$

where (R,θ,ϕ) are the spherical coordinates of the point $x = (x_1,x_2,x_3) \in \Gamma_R$, $u_{mn} = \int_0^{\pi}\int_0^{2\pi}\psi(R,\theta',\phi')\overline{Y_{mn}(\theta',\phi')}\sin\theta'\,d\theta' d\phi'$, $h_n^{(j)}(x)$ are the spherical Hankel functions ($j = 1,2$), $Y_{mn}(\theta,\phi) := \frac{1}{2}\sqrt{\frac{(2n+1)(n-|m|)!}{\pi(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{im\phi}$ are the normalized spherical functions, and $P_n^m(t)$ are the associated Legendre functions ($-n \le m \le n$, $n = 0,1,\ldots,\infty$); and formula (1.18) *should be replaced by*

$$\varphi(r_P,\theta_P,\phi_P) = \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{h_n^{(1)}(kr_P)}{h_n^{(1)}(kR)}\sum_{m=-n}^{n} u_{mn}Y_{mn}(\theta_P,\phi_P), \quad r_P \ge R. \tag{1.187}$$

Remark 4. *The analysis performed in Section 2 enables us to state that all the results of this section remain valid, for example, when in BVP* (1.49)–(1.52) *the Helmholtz equation* (1.50) *is replaced by*

$$-(\Delta + k^2 n(x))\varphi(x) = f(x) \quad in \quad \mathbb{R}^n \setminus \bar{\Omega}, \tag{1.188}$$

where $n = 2, 3$, $k = const > 0$, *the function* $n \in C(\mathbb{R}^n \setminus \Omega)$ *is positive in* $\mathbb{R}^n \setminus \Omega$, *and* $n(x) = 1$ *in the domain* $\Omega_{R_0} \setminus \bar{\Omega}$ *for a certain* $R_0 > 0$.

In this case one has to choose in the equation systems (1.76)–(1.83) *and* (1.114)–(1.121) *which specify minimax mean-square estimates the value of* R *greater than* R_0, *set* $k = \bar{k}$, *and replace in equations* (1.77), (1.81), (1.115), *and* (1.119) k^2 *by* $k^2 n(x)$.

Note also that applying DtN finite-element methods to problems (1.76)–(1.83) and (1.114)–(1.121) one can construct approximate methods of their solution.

PART 2

Minimax estimation of the solutions to the boundary value problems from observations distributed on a system of surfaces. Reduction to a surface integral equation systems

2.1 Statement of the problem

Before to formulate the estimation problem which is the subject of analysis of the present chapter, let us introduce the necessary notations and functional spaces

Let Λ be closed or unclosed $(n-1)$-dimensional Lipschitz surface in $\mathbb{R}^n$. By $d\Lambda$ we will denote the element of measure on surface Λ and by $L^2(\Lambda)$ the space of square integrable functions on Λ.

Let γ be an unclosed $(n-1)$-dimensional smooth C^∞-surface in $\mathbb{R}^n$, $\partial\gamma$ its boundary whose points do not belong to γ, $\partial\gamma\cap\gamma=\emptyset$, and $\hat{\gamma}$ a closed smooth C^∞-surface ($(n-1)$-dimensional manifold without edge) that contains γ, $\gamma\subset\hat{\gamma}$, and divides $\mathbb{R}^n$ into two domains, bounded and unbounded. Set

$$H^{1/2}(\gamma):=\{v|_\gamma:\ v\in H^{1/2}(\hat{\gamma})\}.$$

The norm in space $H^{1/2}(\gamma)$ is determined according to the formula

$$v\in H^{1/2}(\gamma)\Longrightarrow \|v\|_{H^{1/2}(\gamma)}=\inf_{V\in H^{1/2}(\hat{\gamma}),\,V|_\gamma=v}\|V\|_{H^{1/2}(\hat{\gamma})}.$$

Denote by $H^{-1/2}(\gamma)=(H^{1/2}(\gamma))'$ the space dual to $H^{1/2}(\gamma)$. Below, the duality relation $<r,w>_\gamma$ on $H^{-1/2}(\gamma)\times H^{1/2}(\gamma)$ will be also denoted by $\int_\gamma r\bar{w}\,d\gamma$ because for this relation the condition (*) on page 8 is valid in which Γ should be replaced by γ. Note that the elements of $H^{-1/2}(\gamma)$ extended to $\hat{\gamma}\setminus\gamma$ by zero values belong to $H^{-1/2}(\hat{\gamma})$.

Let $\rho(x)$ be a function regular on surface γ which is equivalent to the distance from a point x to the boundary $\partial\gamma$ of γ (this distance will be denoted by $d(x,\partial\gamma)$) in a vicinity[6] of $\partial\gamma$. Following [30] and [50] introduce the space

$$H^{1/2}_{00}(\gamma):=\{u\in H^{1/2}(\gamma),\ \rho^{-1/2}u\in L^2(\gamma)\}=\{u\in H^{1/2}(\gamma),\ \tilde{u}\in H^{1/2}(\hat{\gamma})\}$$

[6]It means that $\lim_{x\to x_0}\frac{\rho(x)}{d(x,\partial\gamma)}=c=\text{const}\neq 0\quad\forall x_0\in\partial\gamma$. Such functions exist because $\partial\gamma$ is an infinitely differentiable manifold [30].

with the norm

$$||u||_{H^{1/2}_{00}(\gamma)} = \left(||u||^2_{H^{1/2}(\gamma)} + ||\rho^{-1/2}u||^2_{L^2(\gamma)}\right)^{1/2},$$

where function $\tilde{u}$ denotes the extension of u by 0 on $\hat{\gamma} \setminus \gamma$.

By $\left(H^{1/2}_{00}(\gamma)\right)'$ we will denote a space conjugate to $H^{1/2}_{00}(\gamma)$. Then, in line with [50], p. 43, we have

$$\left(H^{1/2}_{00}(\gamma)\right)' = \{f = f_0 + f_1,\ f_0 \in H^{-1/2}(\gamma),\ \rho^{1/2} f_1 \in L^2(\gamma)\}.$$

Now let us formulate the estimation problem. Assume that the state $\varphi(x)$ of a system is determined as a solution to the Neumann problem[7]

$$\varphi \in H^1_{\text{loc}}((\mathbb{R}^3 \setminus \bar{\Omega}), \Delta), \tag{2.1}$$

$$(\Delta + k^2)\varphi(x) = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \bar{\Omega}, \tag{2.2}$$

$$\frac{\partial \varphi}{\partial \nu_A} = h \quad \text{on} \quad \Gamma, \tag{2.3}$$

$$\frac{\partial \varphi}{\partial r} - ik\varphi = o(1/r),\ r = |x|,\ r \to \infty, \tag{2.4}$$

where[8] $\Omega \in \mathbb{R}^3$ is a bounded domain with a connected complement such that $\partial\Omega = \Gamma$ is a surface of class C^2, $h \in L^2(\Gamma)$.

It is known that problem (2.1)–(2.4) is uniquely solvable.

Let γ_i, $i = \overline{1, N}$, be smooth simply-connected oriented surfaces in $\mathbb{R}^3$ with smooth boundaries $\partial\gamma_i$, $\partial\gamma_i \cap \gamma_i = \emptyset$, contained in the domain $\mathbb{R}^3 \setminus \bar{\Omega}$ that have no intersections pairwise, $\bar{\gamma}_i \cap \bar{\gamma}_j = \emptyset$, $i \neq j$, $\bar{\gamma}_i \subset \mathbb{R}^3 \setminus \bar{\Omega}$. Let the orientation of γ_i be determined by a continuous family of normals $\nu(x)$, $x \in \gamma_i$.

Assume that on surfaces γ_i the following functions are observed

$$y_i^{(1)}(x) = \int_{\gamma_i} K_i^{(1,1)}(x,y)\varphi(y)\, d\gamma_{i_y} + \int_{\gamma_i} K_i^{(1,2)}(x,y) \frac{\partial \varphi(y)}{\partial \nu}\, d\gamma_{i_y} + \eta_i^{(1)}(x), \tag{2.5}$$

$$y_i^{(2)}(x) = \int_{\gamma_i} K_i^{(2,1)}(x,y)\varphi(y)\, d\gamma_{i_y} + \int_{\gamma_i} K_i^{(2,2)}(x,y) \frac{\partial \varphi(y)}{\partial \nu}\, d\gamma_{i_y} + \eta_i^{(2)}(x), \tag{2.6}$$

---

[7]In this chapter we will restrict ourselves to the case $n = 3$. The results obtained for $n = 3$ remain valid for $n = 2$ after corresponding replacement of Sommerfeld radiation condition and fundamental solution.

[8]It is known (see [40], page 221) that there exists a uniquely determined continuous operator which we denote by $\frac{\partial}{\partial \nu}$ and which maps space $H^1(\Omega, \Delta)$ into space $H^{-1/2}(\Gamma)$ and is such that $\forall u \in H^1(\Omega, \Delta)$, $\forall v \in H^1(\Omega)$ the following representation (Green's formula) holds: $\sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = -\int_\Omega \Delta u\, v\, dx + \int_\Gamma \frac{\partial u}{\partial \nu} v\, d\Gamma$, where the integrals over Γ should be understood as the duality relations $< \frac{\partial u}{\partial \nu}, \mu v >$ on $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. This operator is called the normal derivative in relation to $-\Delta$; the operator $\frac{\partial}{\partial \nu}$ is defined by $\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^n \frac{\partial u}{\partial x_j} \cos(\nu, x_i)$ when $u \in C^\infty(\bar{\Omega})$, where ν is the unit normal vector of Γ external with respect to domain Ω and $\cos(\nu, x_i)$ is the ith directional cosine of ν.

$$x \in \gamma_i, \quad i = \overline{1, N},$$

where $\varphi(x)$ is the solution[9] to BVP (2.1)–(2.3), $\eta_i^{(1)}(x)$ and $\eta_i^{(2)}(x)$ are the observation errors that are choice functions of random fields defined on surfaces γ_i; $K_i^{(r,j)} \in L^2(\gamma_i \times \gamma_i)$, $r, j = 1, 2$, are functions defined on $\gamma_i \times \gamma_i$; and integral operators $G_i^{(j)}$ with kernels $K_i^{(j,2)}(\xi, x)$ defined according to

$$G_i^{(j)}\psi(x) = \int_{\gamma_i} K_i^{(j,2)}(\xi, x)\psi(\xi)d\gamma_{i_\xi}, \quad j = 1, 2, \tag{2.7}$$

are linear bounded operators acting from $L^2(\gamma_i)$ to $H_{00}^{1/2}(\gamma_i)$, $i = \overline{1, N}$ (as an example of such kernels, one may take degenerated kernels $K_i^{(j,2)}(\xi, x) = \sum_{r=1}^{l} a_r^{(ij)}(\xi)b_r^{(ij)}(x)$, where $a_r^{(ij)} \in L^2(\gamma_i)$, $b_r^{(ij)} \in H_{00}^{1/2}(\gamma_i)$).

From the physical viewpoint, observations of the form (2.5), (2.6) enable one, e.g. in stationary problems of hydro acoustics, to observe independently both the pressure and the normal velocity component as well as their linear combinations on a system of surfaces γ_i, $i = \overline{1, N}$.

Denote by G_0 the set of functions $\tilde{h}$, $\tilde{h} \in L^2(\Gamma)$ that satisfy the condition

$$\int_\Gamma |\tilde{h} - h_0|^2 q_1^2 d\Gamma \le 1, \tag{2.8}$$

where $h_0 \in L^2(\Gamma)$ is a given function. By G_1 we denote the set of random vector-functions $\tilde{\eta}(\cdot) = (\tilde{\eta}_1^{(1)}(\cdot), \ldots, \tilde{\eta}_N^{(1)}(\cdot), \tilde{\eta}_1^{(2)}(\cdot), \ldots, \tilde{\eta}_N^{(2)}(\cdot))$; their components $\tilde{\eta}_i^{(1)}(x)$ and $\tilde{\eta}_i^{(2)}(x)$ are random fields defined on surfaces γ_i, $i = \overline{1, N}$ having square integrable second moments and satisfying the conditions

$$\mathbf{E}\tilde{\eta}_i^{(1)}(x) = 0, \quad \mathbf{E}\tilde{\eta}_i^{(2)}(x) = 0, \quad i = \overline{1, N}, \tag{2.9}$$

$$\sum_{i=1}^{N}\int_{\gamma_i} \mathbf{E}|\tilde{\eta}_i^{(1)}(x)|^2 \left(r_i^{(1)}(x)\right)^2 d\gamma_i + \sum_{i=1}^{N}\int_{\gamma_i} \mathbf{E}|\tilde{\eta}_i^{(2)}(x)|^2 \left(r_i^{(2)}(x)\right)^2 d\gamma_i \le 1, \tag{2.10}$$

where $q_1(x), r_i^{(1)}(x), r_i^{(2)}(x)$, $i = \overline{1, N}$, are functions continuous on Γ and $\bar{\gamma}_i$, respectively, that do not vanish on these sets.

Assume also that in equalities (2.1)–(2.3) function $h(x)$ and the second moments $\mathbf{E}|\eta_i^{(1)}(x)|^2$ and $\mathbf{E}|\eta_i^{(2)}(x)|^2$ of random fields $\eta_i^{(1)}(x)$ and $\eta_i^{(2)}(x)$ in observations (2.5) and (2.6) are not known exactly, and it is known only that

$$h \in G_0, \quad \eta(\cdot) = (\eta_1^{(1)}, \ldots, \eta_N^{(1)}(\cdot), \eta_1^{(2)}(\cdot), \ldots, \eta_N^{(2)}(\cdot)) \in G_1. \tag{2.11}$$

[9] Note that for any subdomain ω such that $\bar{\omega} \in \Omega$, the solution φ to problem (2.1)–(2.3) belongs to $H^2(\omega)$; therefore, according to the trace theorem, $\varphi|_{\gamma_i}, \frac{\partial\varphi(y)}{\partial\nu}|_{\gamma_i} \in L^2(\gamma_i)$, $i = \overline{1, N}$, and integrals (2.5)–(2.6) make sense.

Suppose that a function $l_0 \in L^2(\omega_0)$ is defined in a domain ω_0, $\bar{\omega}_0 \subset \mathbb{R}^3 \setminus \bar{\Omega}$. The problem is as follows: given observations (2.5), (2.6) of the state $\varphi(x)$ of a system described by the Neumann BVP (2.1)$-$(2.3) under the conditions that $h \in G_0$ and $\eta \in G_1$, estimate the value of the linear functional

$$l(\varphi) = \int_{\omega_0} \overline{l_0(x)} \varphi(x)\, dx \tag{2.12}$$

in the class of estimates linear with respect to observations that have the form

$$\widehat{l(\varphi)} = \sum_{i=1}^{N} \int_{\gamma_i} \left(\overline{u_i^{(1)}(x)} y_i^{(1)}(x) + \overline{u_i^{(2)}(x)} y_i^{(2)}(x) \right) d\gamma_i + c, \tag{2.13}$$

where $u_i^{(1)}, u_i^{(2)} \in L^2(\gamma_i)$, $i = \overline{1,N}$, $c \in \mathbb{C}$.

Put $u := (u_1^{(1)}, \ldots, u_N^{(1)}, u_1^{(2)}, \ldots, u_N^{(2)}) \in H := \left(L^2(\gamma_1) \times \ldots \times L^2(\gamma_N) \right)^2$.

Definition 2.1. *An estimate*

$$\widehat{\widehat{l(\varphi)}} = \sum_{i=1}^{N} \int_{\gamma_i} \left(\overline{\hat{u}_i^{(1)}(x)} y_i^{(1)}(x) + \overline{\hat{u}_i^{(2)}(x)} y_i^{(2)}(x) \right) d\gamma_i + \hat{c},$$

in which functions $\hat{u}_i^{(1)}$, $\hat{u}_i^{(2)}$ and number $\hat{c}$ are determined from the condition

$$\sup_{\tilde{h} \in G_0,\, \tilde{\eta} \in G_1} \mathbf{E} |l(\tilde{\varphi}) - \widehat{l(\tilde{\varphi})}|^2 \to \inf_{u \in H,\, c \in \mathbb{C}}, \tag{2.14}$$

where

$$\widehat{l(\tilde{\varphi})} = \sum_{i=1}^{N} \int_{\gamma_i} \left(\overline{u_i^{(1)}(x)} \tilde{y}_i^{(1)}(x) + \overline{u_i^{(2)}(x)} \tilde{y}_i^{(2)}(x) \right) d\gamma_i + c, \tag{2.15}$$

$$\tilde{y}_i^{(1)}(x) = \int_{\gamma_i} K_i^{(1,1)}(x,y) \tilde{\varphi}(y)\, d\gamma_{i_y} + \int_{\gamma_i} K_i^{(1,2)}(x,y) \frac{\partial \tilde{\varphi}(y)}{\partial \nu}\, d\gamma_{i_y} + \tilde{\eta}_i^{(1)}(x), \tag{2.16}$$

$$\tilde{y}_i^{(2)}(x) = \int_{\gamma_i} K_i^{(2,1)}(x,y) \tilde{\varphi}(y)\, d\gamma_{i_y} + \int_{\gamma_i} K_i^{(2,2)}(x,y) \frac{\partial \tilde{\varphi}(y)}{\partial \nu}\, d\gamma_{i_y} + \tilde{\eta}_i^{(2)}(x), \tag{2.17}$$

$$x \in \gamma_i, \quad i = \overline{1,N},$$

and $\tilde{\varphi}(x)$ is the solution to the Neumann BVP at $h = \tilde{h}$, will be called a minimax estimate of expression (2.12).

The quantity

$$\sigma := \{ \sup_{\tilde{h} \in G_0,\, \tilde{\eta} \in G_1} \mathbf{E} |l(\tilde{\varphi}) - \widehat{\widehat{l(\tilde{\varphi})}}|^2 \}^{1/2} \tag{2.18}$$

will be called the error of the minimax estimation of $l(\varphi)$.

2.2 Auxiliary statements

In this section we will prove that finding the minimax estimate is equivalent to a certain problem of optimal control of a system described by elliptic equations with conjugation conditions on surfaces γ_i, $i = \overline{1, N}$.

In order to state the conjugation problems under study and prove the existence of their solutions it is necessary to introduce the corresponding Sobolev spaces and trace theorems for surfaces with edges. First, let us formulate several definitions.

For $f \in \mathcal{D}(\mathbb{R}^n)$ the function $u(x) := \int_{\mathbb{R}^n} \Phi_k(x, y) f(y) dy$ solves $-\Delta u - k^2 u = f$ (and complies with Sommerfeld radiation conditions). Here

$$\Phi_k(x, y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x - y|) & \text{in} \ \ \mathbb{R}^2, \\ \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|} & \text{in} \ \ \mathbb{R}^3 \end{cases}$$

is the fundamental solution to the Helmholtz operator.

Introduce the Newton potential operator $(N_k f)(x) := \int_{\mathbb{R}^n} \Phi_k(x, y) f(y) dy$.

Lemma. *(see, for example, [6]) N_k can be extended to a bounded operator N_k : $H^{-1}_{\text{comp}}(\mathbb{R}^n) \to H^1_{\text{loc}}(\mathbb{R}^n)$.*

For $\psi, \chi \in C(\Gamma)$ introduce the functions

$$(\mathcal{V}_\Gamma^k \psi)(x) := \int_\Gamma \Phi_k(x, y) \psi(y) \, d\Gamma_y, \tag{2.19}$$

$$(\mathcal{W}_\Gamma^k \chi)(x) := \int_\Gamma \frac{\partial \Phi_k(x, y)}{\partial \nu_y} \chi(y) \, d\Gamma_y, \qquad x \notin \Gamma, \tag{2.20}$$

called the single and double layer potentials. Let us formulate the results contained in [50] in the following form.

Lemma. *$\mathcal{V}_\Gamma^k$ and $\mathcal{W}_\Gamma^k$ can be extended to bounded operators $\mathcal{V}_\Gamma^k : H^{-1/2}(\Gamma) \to H^1_{\text{loc}}(\mathbb{R}^3) \cap H^1_{\text{loc}}\left(\Omega \cup (\mathbb{R}^3 \setminus \bar{\Omega}), \Delta\right)$ and $\mathcal{W}_\Gamma^k : H^{1/2}(\Gamma) \to H^1_{\text{loc}}\left(\Omega \cup (\mathbb{R}^3 \setminus \bar{\Omega}), \Delta\right)$.*[10]

Let γ_0 be a smooth bounded oriented simply-connected unclosed smooth surface in $\mathbb{R}^3$ with a smooth boundary $\partial\gamma_0$, $\partial\gamma_0 \cap \gamma_0 = \emptyset$. Let its orientation be specified by a continuous family of unit normals $\nu(x)$, $x \in \gamma_0$. Denote by Ω_0 such a bounded open set in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega_0 =: \hat{\gamma}_0$ containing surface γ_0 that the normal

[10] Here, the boundedness of, e.g., the operator $\mathcal{V}_\Gamma^k : H^{-1/2}(\Gamma) \to H^1_{\text{loc}}(\mathbb{R}^3)$ means that for any cutoff function $\alpha \in C^\infty_{\text{comp}}(\mathbb{R}^3)$ the operator $\alpha\mathcal{V}_\Gamma^k : H^{-1/2}(\Gamma) \to H^1(\mathbb{R}^3)$ is bounded.

vector ν to γ_0 is directed outside Ω_0; by γ_{0-} we denote the side of γ_0, whose orientation coincides with that of the external side of surface $\partial\Omega_0$ and by γ_{0+} the opposite side of $\partial\Omega_0$.

Let $\gamma_1, \dots, \gamma_N$ be the surfaces of the type introduced above (see p. 46) with the sides $\gamma_{1+}, \gamma_{1-}, \dots, \gamma_{N+}, \gamma_{N-}$ respectively.

Set $\Omega' = (\mathbb{R}^3 \setminus \bar{\Omega}) \setminus \cup_{i=1}^N \bar{\gamma}_i$ and for any function v defined in Ω' denote by $v|_{\gamma_{i+}}$ and $v|_{\gamma_{i-}}$ the restriction of v to γ_{i+} and to γ_{i-}, $i = \overline{1, N}$, respectively.

Assume that the functions $g, \tilde{g} \in H^0_{\text{comp}}(\Omega') =: L^2_{\text{comp}}(\Omega')$, $\alpha, \tilde{\alpha} \in H^{-1/2}(\Gamma)$, $\omega_i^{(1)}, \varrho_i^{(1)} \in H^{1/2}_{00}(\gamma_i)$, and $\omega_i^{(2)}, \varrho_i^{(2)} \in H^{-1/2}(\gamma_i)$, $i = \overline{1, N}$ are defined in domain Ω'.

Consider two problems.

1. Find function u that satisfies the conditions

$$u \in H^1_{\text{loc}}(\Omega', \Delta), \tag{2.21}$$

$$-(\Delta + k^2)u(x) = g(x) \quad \text{in} \quad \Omega', \tag{2.22}$$

$$\frac{\partial u}{\partial \nu} = \alpha \quad \text{on} \quad \Gamma, \tag{2.23}$$

$$[u]_{\gamma_i} = \omega_i^{(1)} \quad \text{on} \quad \gamma_i, \quad i = \overline{1, N}, \tag{2.24}$$

$$\left[\frac{\partial u}{\partial \nu}\right]_{\gamma_i} = \omega_i^{(2)} \quad \text{on} \quad \gamma_i, \quad i = \overline{1, N}. \tag{2.25}$$

$$\frac{\partial u}{\partial r} - iku = o(1/r), \ r = |x|, \ r \to \infty, \text{ if } \operatorname{Im} k \geq 0. \tag{2.26}$$

Here $[u]_{\gamma_0} = u|_{\gamma_{i+}} - u|_{\gamma_{i-}} \in H^{1/2}_{00}(\gamma_i)$, $u|_{\gamma_{i+}}, u|_{\gamma_{i-}} \in H^{1/2}(\gamma_i)$, $\left[\frac{\partial u}{\partial \nu}\right]_{\gamma_i} = \frac{\partial u}{\partial \nu}\Big|_{\gamma_{i+}} - \frac{\partial u}{\partial \nu}\Big|_{\gamma_{i-}} \in H^{-1/2}(\gamma_i)$, $\frac{\partial u}{\partial \nu}\Big|_{\gamma_{i+}}, \frac{\partial u}{\partial \nu}\Big|_{\gamma_{i-}} \in \left(H^{1/2}_{00}(\gamma_i)\right)'$, $i = \overline{1, N}$, $\frac{\partial u}{\partial \nu}\big|_{\Gamma} \in H^{-1/2}(\Gamma)$, and (2.22)–(2.25) should be understood as equalities of elements from spaces, respectively, $L^2(\Omega')$, $H^{-1/2}(\Gamma)$, $H^{1/2}_{00}(\gamma_i)$, and $H^{-1/2}(\gamma_i)$, $i = \overline{1, N}$.

2. Find function v that satisfies the conditions

$$v \in H^1_{\text{loc}}(\Omega', \Delta), \tag{2.27}$$

$$-(\Delta + \bar{k}^2)v(x) = \tilde{g}(x) \quad \text{in} \quad \Omega', \tag{2.28}$$

$$\frac{\partial v}{\partial \nu} = \tilde{\alpha} \quad \text{on} \quad \Gamma, \tag{2.29}$$

$$[v]_{\gamma_i} = \varrho_i^{(1)} \quad \text{on} \quad \gamma_i, \quad i = \overline{1, N}, \tag{2.30}$$

$$\left[\frac{\partial v}{\partial \nu}\right]_{\gamma_i} = \varrho_i^{(2)} \quad \text{on} \quad \gamma_i, \quad i = \overline{1, N}. \tag{2.31}$$

$$\frac{\partial v}{\partial r} + i\bar{k}u = o(1/r),\ r = |x|,\ r \to \infty, \tag{2.32}$$

where $[v]_{\gamma_0} = v|_{\gamma_{i+}} - v|_{\gamma_{i-}} \in H_{00}^{1/2}(\gamma_i)$, $v|_{\gamma_{i+}}, v|_{\gamma_{i-}} \in H^{1/2}(\gamma_i)$, $\left[\frac{\partial v}{\partial \nu}\right]_{\gamma_i} = \frac{\partial v}{\partial \nu}\Big|_{\gamma_{i+}} - \frac{\partial v}{\partial \nu}\Big|_{\gamma_{i-}} \in H^{-1/2}(\gamma_i)$, $\frac{\partial v}{\partial \nu}\Big|_{\gamma_{i+}}, \frac{\partial v}{\partial \nu}\Big|_{\gamma_{i-}} \in \left(H_{00}^{1/2}(\gamma_i)\right)'$, $i = \overline{1, N}$, $\frac{\partial v}{\partial \nu}\big|_{\Gamma} \in H^{-1/2}(\Gamma)$, and (2.28)–(2.31) should be understood as equalities of elements from the corresponding spaces.

Remark 5. *The choice of spaces for problems 1 and 2 is governed by the trace theorems (see [50], pp. 44, 45 and [40], pp. 180, 181).*

In order to prove the existence and uniqueness of solutions to problems 1 and 2 we formulate one more known result. Namely, let γ_0 and $\hat{\gamma}_0$ be the surfaces introduced on p. 50. By virtue of the definition of spaces $H_{00}^{1/2}(\gamma_0)$ and $H^{-1/2}(\gamma_0)$, the elements of these spaces extended by zero on $\hat{\gamma}_0 \setminus \gamma_0$ are the elements of $H^{1/2}(\hat{\gamma}_0)$ and $H^{-1/2}(\hat{\gamma}_0)$, respectively.

Set

$$X_{\gamma_0}^1 := \{u \in D'(\mathbb{R}^3),\ u \in H^1(\Omega_R \setminus \bar{\gamma}_0),\ \Delta u \in H^1(\Omega_R \setminus \bar{\gamma}_0)\ \forall \text{ sufficiently large } R\}. \tag{2.33}$$

If the tilde sign marks the zero extension on $\hat{\gamma}_0 \setminus \bar{\gamma}_0$ of an element defined on γ_0, the following statement holds.

Theorem 2.1. *Let $\rho \in H_{00}^{1/2}(\gamma_0)$ and $\rho' \in H^{-1/2}(\gamma_0)$. Then $\mathcal{W}_{\gamma_0}^k \rho := \mathcal{W}_{\hat{\gamma}_0}^k \tilde{\rho}$ and $\mathcal{V}_{\gamma_0}^k \rho' := \mathcal{V}_{\hat{\gamma}_0}^k \tilde{\rho}'$ belong to $X_{\gamma_0}^1$, and $\mathcal{V}\rho' \in H^1(\mathbb{R}^3)$.*

Formulate the BVP: find $u \in H_{\mathrm{loc}}^1(\mathbb{R}^3 \setminus \bar{\gamma}_0)$ satisfying

$$u \in H_{\mathrm{loc}}^1(\mathbb{R}^3 \setminus \bar{\gamma}_0), \tag{2.34}$$

$$\Delta u(x) + k^2 u(x) = 0 \text{ in } \mathbb{R}^3 \setminus \bar{\gamma}_0, \tag{2.35}$$

$$[u]_{\gamma_0} = \rho, \quad \left[\frac{\partial u}{\partial \nu}\right]_{\gamma_0} = \rho', \tag{2.36}$$

$$\frac{\partial u}{\partial r} - iku = o(1/r),\ r = |x|,\ r \to \infty. \tag{2.37}$$

The next statement is an immediate corollary of the last theorem.

Theorem 2.2. *Let $\rho \in H^{1/2}_{00}(\gamma_0)$ and $\rho' \in H^{-1/2}(\gamma_0)$. Then BVP (2.34)–(2.37) has a unique solution $u \in X^1_{\gamma_0}$ which, for x outside γ_0, can be given by the formula*

$$u = \mathcal{V}^k_{\gamma_0}\rho' - \mathcal{W}^k_{\gamma_0}\rho. \tag{2.38}$$

Here $[u]_{\gamma_0} = u|_{\gamma_0+} - u|_{\gamma_0-} \in H^{1/2}_{00}(\gamma_0)$, $u|_{\gamma_0+}, u|_{\gamma_0-} \in H^{1/2}(\gamma_0)$, $\left[\frac{\partial u}{\partial \nu_A}\right]_{\gamma_0} = \frac{\partial u}{\partial \nu_A}\Big|_{\gamma_0+} - \frac{\partial u}{\partial \nu_A}\Big|_{\gamma_0-} \in H^{-1/2}(\gamma_0)$, and $\frac{\partial u}{\partial \nu_A}\Big|_{\gamma_0+}$, $\frac{\partial u}{\partial \nu_A}\Big|_{\gamma_0-} \in \left(H^{1/2}_{00}(\gamma_0)\right)'$.

Now let us prove, e.g. for problem 2, that the following statement is valid.

Theorem 2.3. *BVP (2.27)–(2.32) is uniquely solvable and the estimate*

$$\|v\|_{H^1(\Omega'\cap\Omega_R)} \le C_0\Big(\|\tilde{g}\|_{H^{-1}(\Omega_0)} + \|\tilde{\alpha}\|_{H^{-1/2}(\Gamma)} + \sum_{i=1}^N \|\varrho_i^{(1)}\|_{H^{1/2}_{00}(\gamma_i)} + \sum_{i=1}^N \|\varrho_i^{(2)}\|_{H^{-1/2}(\gamma_i)}\Big), \tag{2.39}$$

holds, where Ω_0 is the support of function $\tilde{g}$ and C_0 is a constant which does not depend on $\tilde{g}$, $\tilde{\alpha}$, $\varrho_i^{(1)}$, and $\varrho_i^{(2)}$, $i = \overline{1, N}$.

Proof. Set

$$v_1(x) = \sum_{i=1}^N \mathcal{V}^{-\bar{k}}_{\gamma_i}\varrho_i^{(2)}(x) - \sum_{i=1}^N \mathcal{W}^{-\bar{k}}_{\gamma_i}\varrho_i^{(1)}(x), \quad x \in \mathbb{R}^3 \setminus (\cup_{i=1}^N \bar{\gamma}_i).$$

where, in accordance with definition on page 51, $\mathcal{V}^{-\bar{k}}_{\gamma_i}$ and $\mathcal{W}^{-\bar{k}}_{\gamma_i}$ are single and double layer potentials determined on unclosed surfaces γ_i, $i = 1, N$, corresponding to the wave number $-\bar{k}$. Then by theorem 2.2, function $v_1(x)$ is the unique solution to the problem

$$v_1 \in X^1_{\cup_{i=1}^N \gamma_i}, \tag{2.40}$$

$$-(\Delta + \bar{k}^2)v_1(x) = 0 \quad \text{in} \quad \Omega', \tag{2.41}$$

$$\frac{\partial v_1}{\partial \nu} = 0 \quad \text{on} \quad \Gamma, \tag{2.42}$$

$$[v_1]_{\gamma_i} = \varrho_i^{(1)} \quad \text{on} \quad \gamma_i, \quad i = \overline{1, N}, \tag{2.43}$$

$$\left[\frac{\partial v_1}{\partial \nu}\right]_{\gamma_i} = \varrho_i^{(2)} \quad \text{on} \quad \gamma_i, \quad i = \overline{1, N}. \tag{2.44}$$

$$\frac{\partial v_1}{\partial r} + i\bar{k}v_1 = o(1/r), \ r = |x|, \ r \to \infty, \tag{2.45}$$

Here $[v_1]_{\gamma_0} = v_1|_{\gamma_{i+}} - v_1|_{\gamma_{i-}} \in H^{1/2}_{00}(\gamma_i)$, $v_1|_{\gamma_{i+}}, v_1|_{\gamma_{i-}} \in H^{1/2}(\gamma_i)$, $\left[\frac{\partial v_1}{\partial \nu}\right]_{\gamma_i} = \frac{\partial v_1}{\partial \nu}\Big|_{\gamma_{i+}} - \frac{\partial v_1}{\partial \nu}\Big|_{\gamma_{i-}} \in H^{-1/2}(\gamma_i)$, $\frac{\partial v_1}{\partial \nu}\Big|_{\gamma_{i+}}, \frac{\partial v_1}{\partial \nu}\Big|_{\gamma_{i-}} \in \left(H^{1/2}_{00}(\gamma_i)\right)'$, $i = \overline{1, N}$, and $\frac{\partial v_1}{\partial \nu}\Big|_{\Gamma} \in H^{-1/2}(\Gamma)$. Denote by v_2 the unique solution to the problem

$$v_2 \in H^1_{\mathrm{loc}}\left((\mathbb{R}^3 \setminus \bar{\Omega}), \Delta\right) \tag{2.46}$$

$$-(\Delta + \bar{k}^2)v_2(x) = 0, \quad x \in \mathbb{R}^3 \setminus \bar{\Omega}, \tag{2.47}$$

$$\frac{\partial v_2}{\partial \nu} = -\frac{\partial v_1}{\partial \nu} \quad \text{on} \quad \Gamma, \tag{2.48}$$

$$\frac{\partial v_2}{\partial r} + i\bar{k}v_2 = o(1/r),\ r = |x|,\ r \to \infty, \tag{2.49}$$

and by v_3 the unique solution to the problem

$$v_3 \in H^1_{\mathrm{loc}}\left((\mathbb{R}^3 \setminus \bar{\Omega}), \Delta\right) \tag{2.50}$$

$$-(\Delta + \bar{k}^2)v_3(x) = \tilde{g}, \quad x \in \mathbb{R}^3 \setminus \bar{\Omega}, \tag{2.51}$$

$$\frac{\partial v_3}{\partial \nu} = \tilde{\alpha} \quad \text{on} \quad \Gamma, \tag{2.52}$$

$$\frac{\partial v_3}{\partial r} + i\bar{k}v_3 = o(1/r),\ r = |x|,\ r \to \infty. \tag{2.53}$$

Then the function

$$v(x) = v_1(x) + v_2(x) + v_3(x) \tag{2.54}$$

will be the unique solution to problem (2.28)–(2.32). From equalities $\mathcal{V}^{-\bar{k}}_{\gamma_i}\varrho^{(2)}_i(x) = \mathcal{V}^{-\bar{k}}_{\hat{\gamma}_i}\tilde{\varrho}^{(2)}_i(x)$ and $\mathcal{W}^{-\bar{k}}_{\gamma_i}\varrho^{(2)}_i(x) = \mathcal{W}^{-\bar{k}}_{\hat{\gamma}_i}\tilde{\varrho}^{(2)}_i(x)$, and boundedness of operstors $\mathcal{V}^{-\bar{k}}_{\hat{\gamma}_i}$ and $\mathcal{W}^{-\bar{k}}_{\hat{\gamma}_i}$ in the corresponding spaces, we obtain the following estimates for v_1 [11]

$$\|v_1\|_{H^1(\Omega' \cap \Omega_R, \Delta)} \le C_1 \sum_{i=1}^{N} \|\tilde{\varrho}^{(1)}_i\|_{H^{1/2}(\hat{\gamma}_i)}, \quad \|v_1\|_{H^1(\Omega' \cap \Omega_R, \Delta)} \le C_2 \sum_{i=1}^{N} \|\tilde{\varrho}^{(2)}_i\|_{H^{-1/2}(\hat{\gamma}_i)}$$

and hence

$$\|v_1\|_{H^1(\Omega' \cap \Omega_R, \Delta)} \le C_3 \left(\sum_{i=1}^{N} \|\varrho^{(1)}_i\|_{H^{1/2}_{00}(\gamma_i)} + 2\sum_{i=1}^{N} \|\varrho^{(2)}_i\|_{H^{-1/2}(\gamma_i)} \right). \tag{2.55}$$

For functions v_2 and v_3 we apply estimate (1.34) to obtain

$$\|v_2\|_{H^1(\Omega_R \setminus \bar{\Omega})} \le C_4 \|\gamma_N v_1\|_{H^{-1/2}(\Gamma)}, \tag{2.56}$$

[11] Here and below C_i are constants that do not depend on the data of the problems in question.

$$\|v_3\|_{H^1(\Omega_R\setminus\bar{\Omega})} \le C_5 \left(\|\tilde{g}\|_{H^{-1}(\Omega_R\setminus\bar{\Omega})} + \|\tilde{\alpha}\|_{H^{-1/2}(\Gamma)}\right). \tag{2.57}$$

Using the trace theorem ([40], pp. 180, 181) and (2.55), we prove the estimates

$$\|\gamma_N v_1\|_{H^{-1/2}(\Gamma)} \le C_5 \|v_1\|_{H^1(\Omega'\cap\Omega_R,\Delta)} \le C_6 \left(\sum_{i=1}^{N} \|\varrho_i^{(1)}\|_{H_{00}^{1/2}(\gamma_i)} + \sum_{i=1}^{N} \|\varrho_i^{(2)}\|_{H^{-1/2}(\gamma_i)}\right). \tag{2.58}$$

From (2.54)–(2.58) it follows that

$$\|v\|_{H^1(\Omega'\cap\Omega_R)} \le C_0 \Big(\|\tilde{g}\|_{H^{-1}(\Omega'\cap\Omega_R)} + \|\tilde{\alpha}\|_{H^{-1/2}(\Gamma)} + \sum_{i=1}^{N} \|\varrho_i^{(1)}\|_{H_{00}^{1/2}(\gamma_i)} + \sum_{i=1}^{N} \|\varrho_i^{(2)}\|_{H^{-1/2}(\gamma_i)}\Big).$$

□

For problem 1 the corresponding theorem is obtained in a similar manner.

2.3 General form of the guaranteed estimates and expression for the estimation error

Introduce, for every fixed $u \in H = \left(L^2(\gamma_1) \times \ldots \times L^2(\gamma_N)\right)^2$ the function $z(x;u)$ as a solution to the problem

$$z \in H^1_{\text{loc}}(\Omega', \Delta), \tag{2.59}$$

$$-(\Delta + \bar{k}^2) z(x;u) = \chi_{\omega_0}(x) l_0(x) \quad \text{in} \quad \Omega', \tag{2.60}$$

$$\frac{\partial z}{\partial \nu_{A^*}} = 0 \quad \text{on} \quad \Gamma, \tag{2.61}$$

$$\begin{aligned} [z(x;u)]_{\gamma_i} &= \int_{\gamma_i} \left[\overline{K_i^{(1,2)}(\xi,x)} u_i^{(1)}(\xi) + \overline{K_i^{(2,2)}(\xi,x)} u_i^{(2)}(\xi)\right] d\gamma_{i_\xi}, \\ \left[\frac{\partial z(x;u)}{\partial \nu_{A^*}}\right]_{\gamma_i} &= -\int_{\gamma_i} \left[\overline{K_i^{(1,1)}(\xi,x)} u_i^{(1)}(\xi) + \overline{K_i^{(2,1)}(\xi,x)} u_i^{(2)}(\xi)\right] d\gamma_{i_\xi} \end{aligned} \tag{2.62}$$

$$\text{on} \quad \gamma_i, \quad i = \overline{1,N}.$$

$$\frac{\partial z(x;u)}{\partial r} + i\bar{k} z(x;u) = o(1/r), \ r = |x|, \ r \to \infty. \tag{2.63}$$

Lemma 2.1. *Finding the minimax estimate of $l(\varphi)$ is equivalent to the problem of optimal control of the system described by BVP (2.59)–(2.63) with the cost function*

$$I(u_1^{(1)}, \ldots, u_N^{(1)}, u_1^{(2)}, \ldots, u_N^{(2)}) = \int_{\Gamma} q_1^{-2}(x) z^2(x;u)\, d\Gamma$$

$$+\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(1)}(x))^{-2}(u_i^{(1)}(x))^2 d\gamma_i+\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(2)}(x))^{-2}(u_i^{(2)}(x))^2 d\gamma_i \to \min_{u\in H}. \qquad (2.64)$$

Proof. Denote by Ω_i an open subdomain in $\mathbb{R}^3\setminus\bar{\Omega}$ ($\bar{\Omega}_i \subset \mathbb{R}^3\setminus\bar{\Omega}$) such that $\partial\Omega_i$ contains γ_i, its boundary $\partial\Omega_i$ is simply-connected and smooth, and the normal vector ν to γ_i is directed outside Ω_i. We also assume that $\Omega_i\cap\Omega_j=\emptyset$ for all $i\neq j$, $i,j=\overline{1,N}$.

Set $\tilde{\Omega}_R := (\Omega_R\setminus\bar{\Omega})\setminus\cup_{i=1}^{N}\bar{\Omega}_i$ (R is assumed to be sufficiently large), $\hat{\gamma}_i=\partial\Omega_i$ and denote by $\hat{\gamma}_{i-}$ and $\hat{\gamma}_{i+}$ the external and internal sides of surface $\hat{\gamma}_i$. Next, simplifying the notation in the surface integrals, denote by z_+, $\left(\frac{\partial z}{\partial\nu_{A^*}}\right)_+$ and z_-, $\left(\frac{\partial z}{\partial\nu_{A^*}}\right)_-$ the traces $z|_{\gamma_{k+}}$, $\left.\frac{\partial z}{\partial\nu_{A^*}}\right|_{\gamma_{k+}}$ (or $z|_{\hat{\gamma}_{k+}}$, $\left.\frac{\partial z}{\partial\nu_{A^*}}\right|_{\hat{\gamma}_{k+}}$) and $z|_{\gamma_{k-}}$, $\left.\frac{\partial z}{\partial\nu_{A^*}}\right|_{\gamma_{k-}}$ (or $z|_{\hat{\gamma}_{k-}}$, $\left.\frac{\partial u}{\partial\nu_{A^*}}\right|_{\hat{\gamma}_{k-}}$) of functions $z(x;u)$ or $\frac{\partial z(x;u)}{\partial\nu_{A^*}}$ on sides γ_{k+} (or $\hat{\gamma}_{k+}$) and γ_{k-} (or $\hat{\gamma}_{k-}$) of surface γ_k (or $\hat{\gamma}_k$).

Taking into consideration relationships (2.15)−(2.17), (2.59)−(2.63), and applying to $\tilde{\varphi}(x)$ and $z(x;u)$ in domains Ω_i, $i=1,N$, and $\tilde{\Omega}_R$ the second Green formula[12], we obtain, using the equalities $[z(\cdot;u)]_{\hat{\gamma}_i\setminus\gamma_i}=\left[\frac{\partial z(\cdot;u)}{\partial\nu}\right]_{\hat{\gamma}_i\setminus\bar{\gamma}_i}=0$,

$$l(\tilde{\varphi})-\widehat{l(\tilde{\varphi})}=$$

$$=\int_{\omega_0}\overline{l_0(x)}\tilde{\varphi}(x)\,dx-\sum_{i=1}^{N}\int_{\gamma_{ix}}\left(\overline{u_i^{(1)}(x)}\tilde{y}_i^{(1)}(x)+\overline{u_i^{(2)}(x)}\tilde{y}_i^{(2)}(x)\right)d\gamma_i-c$$

$$=\int_{\tilde{\Omega}_R}\chi_{\omega_0}(x)\overline{l_0(x)}\tilde{\varphi}(x)\,dx+\sum_{i=1}^{N}\int_{\Omega_i}\chi_{\omega_0}(x)\overline{l_0(x)}\tilde{\varphi}(x)\,dx$$

$$-\sum_{i=1}^{N}\int_{\gamma_{ix}}\left(\overline{u_i^{(1)}(x)}\tilde{y}_i^{(1)}(x)+\overline{u_i^{(2)}(x)}\tilde{y}_i^{(2)}(x)\right)d\gamma_i-c$$

$$=-\int_{\tilde{\Omega}_R}\tilde{\varphi}(x)\overline{(\Delta+\bar{k}^2)z(x;u)}\,dx-\sum_{i=1}^{N}\int_{\Omega_i}\tilde{\varphi}(x)\overline{(\Delta+\bar{k}^2)z(x;u)}\,dx$$

$$-\sum_{i=1}^{N}\int_{\gamma_i}\overline{u_i^{(1)}(x)}\left[\int_{\gamma_i}K_i^{(1,1)}(x,\xi)\tilde{\varphi}(\xi)\,d\gamma_{i_\xi}+\int_{\gamma_i}K_i^{(1,2)}(x,\xi)\frac{\partial\tilde{\varphi}(\xi)}{\partial\nu_{A^*}}\,d\gamma_{i_\xi}\right]d\gamma_{i_x}$$

$$-\sum_{i=1}^{N}\int_{\gamma_i}\overline{u_i^{(2)}(x)}\left[\int_{\gamma_i}K_i^{(2,1)}(x,\xi)\tilde{\varphi}(\xi)\,d\gamma_{i_\xi}+\int_{\gamma_i}K_i^{(2,2)}(x,\xi)\frac{\partial\tilde{\varphi}(\xi)}{\partial\nu_{A^*}}\,d\gamma_{i_\xi}\right]d\gamma_{i_x}$$

[12]One can apply the second Green formula because $z(\cdot;u)$, $\varphi\in H^1(\tilde{\Omega}_R)$, $z(\cdot;u)$, $\varphi\in H^1(\Omega_i)$, $i=\overline{1,N}$, and Δz, $\Delta\varphi\in L^2(\tilde{\Omega}_R)$, Δz, $\Delta\varphi\in L^2(\Omega_i)$, $i=\overline{1,N}$,

$$
\begin{aligned}
&-\sum_{i=1}^{N}\int_{\gamma_i}\overline{u_i^{(1)}(x)}\tilde{\eta}_i^{(1)}(x)\,d\gamma_i-\sum_{i=1}^{N}\int_{\gamma_i}\overline{u_i^{(2)}(x)}\tilde{\eta}_i^{(2)}(x)\,d\gamma_i-c\\
&=-\int_{\tilde{\Omega}_R}(\Delta+k^2)\tilde{\varphi}(x)\overline{z(x;u)}\,dx+\int_{\Gamma}\overline{z}\frac{\partial\tilde{\varphi}}{\partial\nu}\,d\Gamma-\sum_{i=1}^{N}\int_{\Omega_i}(\Delta+k^2)\tilde{\varphi}(x)\overline{z(x;u)}\,dx\\
&-\sum_{i=1}^{N}\int_{\hat{\gamma}_i}\left(\overline{z_-(x;u)}\frac{\partial\tilde{\varphi}(x)}{\partial\nu}-\tilde{\varphi}(x)\overline{\left(\frac{\partial z(x;u)}{\partial\nu}\right)_-}\right)\,d\hat{\gamma}_i\\
&+\sum_{i=1}^{N}\int_{\hat{\gamma}_i}\left(\overline{z_+(x;u)}\frac{\partial\tilde{\varphi}(x)}{\partial\nu}-\tilde{\varphi}(x)\overline{\left(\frac{\partial z(x;u)}{\partial\nu}\right)_+}\right)\,d\hat{\gamma}_i+\Sigma_R(z(\cdot;u),\tilde{\varphi})\\
&-\sum_{i=1}^{N}\int_{\gamma_i}\tilde{\varphi}(\xi)\int_{\gamma_i}\left[K_i^{(1,1)}(x,\xi)\overline{u_i^{(1)}(x)}+K_i^{(2,1)}(x,\xi)\overline{u_i^{(2)}(x)}\right]\,d\gamma_{i_x}\,d\gamma_{i_\xi}\\
&-\sum_{i=1}^{N}\int_{\gamma_i}\frac{\partial\tilde{\varphi}(\xi)}{\partial\nu}\int_{\gamma_i}\left[K_i^{(1,2)}(x,\xi)\overline{u_i^{(1)}(x)}+K_i^{(2,2)}(x,\xi)\overline{u_i^{(2)}(x)}\right]\,d\gamma_{i_x}\,d\gamma_{i_\xi}\\
&-\sum_{i=1}^{N}\int_{\gamma_i}\overline{u_i^{(1)}(x)}\tilde{\eta}_i^{(1)}(x)\,d\gamma_i-\sum_{i=1}^{N}\int_{\gamma_i}\overline{u_i^{(2)}(x)}\tilde{\eta}_i^{(2)}(x)\,d\gamma_i-c\\
&=\int_{\Gamma}\overline{z}\tilde{h}\,d\Gamma+\sum_{i=1}^{N}\int_{\gamma_i}(\overline{z_+(x,u)}-\overline{z_-(x,u)})\frac{\partial\tilde{\varphi}(x)}{\partial\nu}\,d\hat{\gamma}_i\\
&-\sum_{i=1}^{N}\int_{\gamma_i}\left[\overline{\left(\frac{\partial z(x,u)}{\partial\nu}\right)_+}-\overline{\left(\frac{\partial z(x,u)}{\partial\nu}\right)_-}\right]\tilde{\varphi}(x)\,d\hat{\gamma}_i\\
&-\sum_{i=1}^{N}\int_{\gamma_i}\tilde{\varphi}(x)\int_{\gamma_i}\left[K_i^{(1,1)}(\xi,x)\overline{u_i^{(1)}(\xi)}+K_i^{(2,1)}(\xi,x)\overline{u_i^{(2)}(\xi)}\right]\,d\gamma_{i_\xi}\,d\gamma_{i_x}+\Sigma_R(z(\cdot;u),\tilde{\varphi})\\
&-\sum_{i=1}^{N}\int_{\gamma_i}\frac{\partial\tilde{\varphi}(x)}{\partial\nu}\int_{\gamma_i}\left[K_i^{(1,2)}(\xi,x)\overline{u_i^{(1)}(\xi)}+K_i^{(2,2)}(\xi,x)\overline{u_i^{(2)}(\xi)}\right]\,d\gamma_{i_\xi}\,d\gamma_{i_x}\\
&-\sum_{i=1}^{N}\int_{\gamma_i}\overline{u_i^{(1)}(x)}\tilde{\eta}_i^{(1)}(x)\,d\gamma_i-\sum_{i=1}^{N}\int_{\gamma_i}\overline{u_i^{(2)}(x)}\tilde{\eta}_i^{(2)}(x)\,d\gamma_i-c\\
&=\int_{\Gamma}\tilde{h}\overline{z}\,d\Gamma+\Sigma_R(z(\cdot;u),\tilde{\varphi})\\
&-\sum_{i=1}^{N}\int_{\gamma_i}\overline{u_i^{(1)}(x)}\tilde{\eta}_i^{(1)}(x)\,d\gamma_i-\sum_{i=1}^{N}\int_{\gamma_i}\overline{u_i^{(2)}(x)}\tilde{\eta}_i^{(2)}(x)\,d\gamma_i-c, \qquad (2.65)
\end{aligned}
$$

where by $\Sigma_R(z(\cdot;u),\tilde{\varphi})$ we denote

$$\Sigma_R(z(\cdot;u),\tilde{\varphi}) := \int_{\Gamma_R}\left(\overline{z(x;u)}\frac{\partial\tilde{\varphi}(x)}{\partial\nu} - \tilde{\varphi}(x)\overline{\left(\frac{\partial z(x;u)}{\partial\nu}\right)}\right)\,d\Gamma_R$$

Since $z(\cdot;u)$ and $\tilde{\varphi}$ satisfy, respectively, the Sommerfeld radiation conditions (2.63) and (2.4) we obtain an estimate for $\Sigma_R(z(\cdot;u),\tilde{\varphi})$,

$$\Sigma_R(z(\cdot;u),\tilde{\varphi}) = \int_{\Gamma_R}\overline{z(x;u)}\left(\frac{\partial\tilde{\varphi}(x)}{\partial R} - ik\varphi(x)\right)\,d\Gamma_R$$
$$- \int_{\Gamma_R}\tilde{\varphi}(x)\overline{\left(\frac{\partial z(x;u)}{\partial R} + i\bar{k}z(x;u)\right)}\,d\Gamma_R$$

$$= \int_{\Gamma_R} O(1/R)o(1/R)d\Gamma_R - \int_{\Gamma_R} O(1/R)o(1/R)d\Gamma_R = o(1) \quad \text{при} \quad R\to\infty.$$

From here, passing to the limit as $R\to\infty$ in (2.65), we obtain

$$l(\tilde{\varphi}) - \widehat{l(\tilde{\varphi})} = \int_\Gamma \tilde{h}\overline{z(\cdot;u)}\,d\Gamma$$
$$-\sum_{i=1}^N\int_{\gamma_i}\overline{u_i^{(1)}(x)}\tilde{\eta}_i^{(1)}(x)\,d\gamma_i - \sum_{i=1}^N\int_{\gamma_i}\overline{u_i^{(2)}(x)}\tilde{\eta}_i^{(2)}(x)\,d\gamma_i - c,$$

The latter equalities together with conditions (2.9)–(2.10) and the known relation $\mathbf{D}\xi = \mathbf{E}\xi^2 - |\mathbf{E}\xi|^2$ that couples dispersion $\mathbf{D}\xi$ of random variable ξ and its expectation $\mathbf{E}\xi$, yield

$$\inf_{c\in\mathbb{C}}\sup_{\tilde{h}\in G_0,\,\tilde{\eta}\in G_1}\mathbf{E}|l(\tilde{\varphi}) - \widehat{l(\tilde{\varphi})}|^2 =$$
$$= \inf_{c\in\mathbb{C}}\sup_{\tilde{h}\in G_0}\left|\int_\Omega\int_\Gamma \tilde{h}\overline{z(\cdot;u)}\,d\Gamma - c\right|^2$$
$$+ \sup_{\tilde{\eta}\in G_1}\mathbf{E}\left|\sum_{i=1}^N\int_{\gamma_i}\overline{u_i^{(1)}(x)}\tilde{\eta}_i^{(1)}(x)\,d\gamma_i + \sum_{i=1}^N\int_{\gamma_i}\overline{u_i^{(2)}(x)}\tilde{\eta}_i^{(2)}(x)\,d\gamma_i\right|^2. \tag{2.66}$$

In order to calculate the supremum in the right-hand side of (2.66) make use of the Cauchy–Bunyakovsky inequality. Introducing the notation

$$y = \int_\Gamma(\tilde{h} - h_0)\overline{z(\cdot;u)}\,d\Gamma,$$

we prove, using relation (2.8), the inequality

$$|y| \le \left\{\int_\Gamma q_1^{-2}(x)|z(x;u)|^2\,d\Gamma\right\}^{\frac{1}{2}} \times \left\{\int_\Gamma|\tilde{h} - h_0|^2 q_1^2 d\Gamma\right\}^{\frac{1}{2}}$$

$$\leq \left\{ \int_\Gamma q_1^{-2}(x)|z(x;u)|^2\,d\Gamma \right\}^{\frac{1}{2}} := a,$$

in which the equality holds at $\tilde h \in G_0$ and

$$\tilde h(\cdot) = \pm \frac{q_1^{-2}(\cdot) z(\cdot;u)\,|_\Gamma}{a} + h_0. \tag{2.67}$$

Therefore,

$$\inf_{c\in\mathbb{C}} \sup_{\tilde h \in G_0} \left| \int_\Gamma \tilde h \overline{z(\cdot;u)}\,d\Gamma - c \right|^2$$

$$= \inf_{c\in\mathbb{C}} \sup_{|y|\leq a} |y + \int_\Gamma \overline{z(\cdot;u)} h_0\,d\Gamma - c|^2 = a^2 = \int_\Gamma q_1^{-2}(x)|z(x;u)|^2\,d\Gamma$$

at

$$c = \int_\Gamma \overline{z(\cdot;u)} h_0\,d\Gamma. \tag{2.68}$$

Similarly,

$$\sup_{\tilde\eta\in G_1} \mathbf{E}\left| \sum_{i=1}^N \int_{\gamma_i} \overline{u_i^{(1)}(x)}\tilde\eta_i^{(1)}(x)\,d\gamma_i + \sum_{i=1}^N \int_{\gamma_i} \overline{u_i^{(2)}(x)}\tilde\eta_i^{(2)}(x)\,d\gamma_i \right|^2$$

$$\leq \sum_{i=1}^N \int_{\gamma_i} (r_i^{(1)}(x))^{-2}|u_i^{(1)}(x)|^2 d\gamma_i + \sum_{i=1}^N \int_{\gamma_i} (r_i^{(2)}(x))^{-2}|u_i^{(2)}(x)|^2 d\gamma_i.$$

It is easy to see that in this inequality, the equality holds when $\tilde\eta$ is a random vector-function with the components

$$\tilde\eta_i^{(1)}(x) = \frac{\xi\,(r_i^{(1)}(x))^{-2}u_i^{(1)}(x)}{\left[\sum_{i=1}^N \int_{\gamma_i}(r_i^{(1)}(x))^{-2}|u_i^{(1)}(x)|^2 d\gamma_i + \sum_{i=1}^N \int_{\gamma_i}(r_i^{(2)}(x))^{-2}|u_i^{(2)}(x)|^2 d\gamma_i\right]^{1/2}},$$

$$\tilde\eta_i^{(2)}(x) = \frac{\xi\,(r_i^{(2)}(x))^{-2}u_i^{(2)}(x)}{\left[\sum_{i=1}^N \int_{\gamma_i}(r_i^{(1)}(x))^{-2}|u_i^{(1)}(x)|^2 d\gamma_i + \sum_{i=1}^N \int_{\gamma_i}(r_i^{(2)}(x))^{-2}|u_i^{(2)}(x)|^2 d\gamma_i\right]^{1/2}}, \tag{2.69}$$

where ξ is a random value such that $\mathbf{E}\xi = 0$ and $\mathbf{E}\xi^2 = 1$. The latter facts yield

$$\inf_{c\in\mathbb{C}} \sup_{\tilde h\in G_0,\, \tilde\eta\in G_1} \mathbf{E}|l(\tilde\varphi) - \widehat{l(\tilde\varphi)}|^2 = I(u_1^{(1)},\ldots,u_{m_1}^{(1)},u_1^{(2)},\ldots,u_{m_2}^{(2)}),$$

where functional I is determined according to (2.64) and the infimum with respect to c is attained at $c = \int_\Gamma \overline{z(\cdot;u)} h_0\,d\Gamma$. The lemma is proved. □

Solving the optimal control problem (2.59)–(2.64), we arrive at the following statement.

Theorem 2.4. *The minimax estimate of the value of functional $l(\varphi)$ has the form*

$$\widehat{\widehat{l(\varphi)}} = \sum_{i=1}^{N} \int_{\gamma_i} \left(\overline{\hat{u}_i^{(1)}(x)} y_i^{(1)}(x) + \overline{\hat{u}_i^{(2)}(x)} y_i^{(2)}(x) \right) d\gamma_i + \hat{c}, \tag{2.70}$$

where

$$\hat{c} = \int_{\Gamma} \overline{z} h_0 \, d\Gamma, \tag{2.71}$$

$$\hat{u}_i^{(1)}(x) = (r_i^{(1)}(x))^2 \int_{\gamma_i} \left[K_i^{(1,1)}(x,\xi) p(\xi) + K_i^{(1,2)}(x,\xi) \frac{\partial p(\xi)}{\partial \nu} \right] d\gamma_{i_\xi}, \tag{2.72}$$
$$\hat{u}_i^{(2)}(x) = (r_i^{(2)}(x))^2 \int_{\gamma_i} \left[K_i^{(2,1)}(x,\xi) p(\xi) + K_i^{(2,2)}(x,\xi) \frac{\partial p(\xi)}{\partial \nu} \right] d\gamma_{i_\xi}, \quad i = \overline{1,N},$$

and function $p(x)$ is determined from the solution to the problem

$$z \in H^1_{\text{loc}}(\Omega', \Delta), \tag{2.73}$$

$$-(\Delta + \bar{k}^2) z(x) = \chi_{\omega_0}(x) l_0(x) \quad in \quad \Omega', \tag{2.74}$$

$$\frac{\partial z}{\partial \nu} = 0 \quad on \quad \Gamma, \tag{2.75}$$

$$[z(x)]_{\gamma_i} = \int_{\gamma_i} \left[\overline{K_i^{(1,2)}(\xi,x)} \hat{u}_i^{(1)}(\xi) + \overline{K_i^{(2,2)}(\xi,x)} \hat{u}_i^{(2)}(\xi) \right] d\gamma_{i_\xi}, \tag{2.76}$$

$$\left[\frac{\partial z(x)}{\partial \nu} \right]_{\gamma_i} =$$
$$= - \int_{\gamma_i} \left[\overline{K_i^{(1,1)}(\xi,x)} \hat{u}_i^{(1)}(\xi) + \overline{K_i^{(2,1)}(\xi,x)} \hat{u}_i^{(2)}(\xi) \right] d\gamma_{i_\xi} \text{ on } \gamma_i, \; i = \overline{1,N}, \tag{2.77}$$

$$\frac{\partial z(x)}{\partial r} + i\bar{k} z(x) = o(1/r), \; r = |x|, \; r \to \infty, \tag{2.78}$$

$$p \in H^1_{\text{loc}}(\Omega', \Delta), \tag{2.79}$$

$$(\Delta + k^2) p(x) = 0 \quad in \quad \Omega', \tag{2.80}$$

$$\frac{\partial p}{\partial \nu} = q_1^{-2} z \quad on \quad \Gamma, \tag{2.81}$$

$$[p]_{\gamma_i} = 0, \quad \left[\frac{\partial p}{\partial \nu} \right]_{\gamma_i} = 0, \quad i = \overline{1,N}. \tag{2.82}$$

$$\frac{\partial p(x)}{\partial r} - ikp(x) = o(1/r),\ r = |x|,\ r \to \infty. \tag{2.83}$$

where in (2.72) $p(y)$ and $\frac{\partial p(y)}{\partial \nu}$ denote the values of the traces of functions p and boundary values of its conormal derivatives on different sides of surface γ_i. Also, $p|_{\gamma_i} \frac{\partial p(y)}{\partial \nu}|_{\gamma_i} \in L^2(\gamma_i)$, $i = \overline{1, N}$. Problem (2.72)–(2.83) is uniquely solvable.

The error σ of the minimax estimation of $l(\varphi)$ is given by the formula

$$\sigma = [l(p)]^{1/2} = \left(\int_{\omega_0} \overline{l_0(x)} p(x)\, dx \right)^{1/2}. \tag{2.84}$$

Note that if we replace in (2.76) and (2.77) $\hat{u}_i^{(1)}(x)$ and $\hat{u}_i^{(2)}(x)$, $i = \overline{1, N}$, by their expressions in the right-hand sides of (2.72), then these functions may be excluded from the equality system (2.73)–(2.83).

Proof. Let us show that $I(u)$ is a quadratic function on H. Indeed, since the solution $z(x; u)$ to problem (2.59)–(2.62) can be represented as $z(x; u) = \tilde{z}(x; u) + z_0(x)$, where $\tilde{z}(x; u)$ is the solution to this problem at $l_0(x) \equiv 0$ and $\tilde{z}_0(x)$ is the solution to the problem

$$z_0 \in H^1_{\text{loc}}(\Omega', \Delta), \tag{2.85}$$

$$-(\Delta + \bar{k}^2) z_0(x) = \chi_{\omega_0}(x) l_0(x) \quad \text{in} \quad \Omega', \tag{2.86}$$

$$\frac{\partial z_0}{\partial \nu} = 0 \quad \text{on} \quad \Gamma, \tag{2.87}$$

$$[z_0(x)]_{\gamma_i} = 0 \quad \text{on} \quad \gamma_i, \quad i = \overline{1, N},$$

$$\left[\frac{\partial z_0(x)}{\partial \nu} \right]_{\gamma_i} = 0 \quad \text{on} \quad \gamma_i, \quad i = \overline{1, N}, \tag{2.88}$$

$$\frac{\partial z_0(x)}{\partial r} + i\bar{k} z_0(x) = o(1/r),\ r = |x|,\ r \to \infty, \tag{2.89}$$

functional $I(u)$ can be represented as

$$I(u) = \tilde{I}(u) + L(u) + \int_\Gamma q_1^{-2} |z_0|^2\, d\Gamma,$$

where

$$\tilde{I}(u) = \int_\Gamma q_1^{-2}(x) |\tilde{z}^2(x; u)|\, d\Gamma$$

$$+ \sum_{i=1}^N \int_{\gamma_i} (r_i^{(1)}(x))^{-2} |u_i^{(1)}(x)|^2 d\gamma_i + \sum_{i=1}^N \int_{\gamma_i} (r_i^{(2)}(x))^{-2} |u_i^{(2)}(x)|^2 d\gamma_i,$$

$$L(u) = 2 \ \mathrm{Re} \int_\Gamma q_1^{-2}(x)\tilde{z}(x;u)\overline{z_0(x)}\, d\Gamma.$$

From inequality (2.39) and our assumptions concerning operators of the form (2.7), we deduce, taking into account that $\|\cdot\|_{H^{-1/2}(\gamma_i)} \le c\|\cdot\|_{L^2(\gamma_i)}$ ($c = \mathrm{const} > 0$), the inequality

$$\|\tilde{z}(\cdot;u)\|_{H^1(\Omega'_R)} \le \Big(\sum_{i=1}^N \Big\| \int_{\gamma_i} \Big[\overline{K_i^{(1,2)}(\xi,x)} u_i^{(1)}(\xi) + \overline{K_i^{(2,2)}(\xi,x)} u_i^{(2)}(\xi)\Big]\, d\gamma_{i_\xi}\Big\|^2_{H^{1/2}_{00}(\gamma_i)}\Big)^{1/2}$$

$$+\Big(\sum_{i=1}^N \Big\| \int_{\gamma_i} \Big[\overline{K_i^{(1,1)}(\xi,x)} u_i^{(1)}(\xi) + \overline{K_i^{(2,1)}(\xi,x)} u_i^{(2)}(\xi)\Big]\, d\gamma_{i_\xi}\Big\|^2_{H^{-1/2}(\gamma_i)}\Big)^{1/2} \le c_1\|u\|_H, \tag{2.90}$$

where $c_1 = \mathrm{const}$ that does not depend on u. Taking into account (2.90) and the trace theorem from [1], we see that $u \to \gamma_D\tilde{z}(\cdot;u)$ is a bounded linear operator that maps Hilbert space H in $H^{1/2)}(\Gamma)$. From the latter statement, it follows that $\tilde{I}(u)$ is a quadratic form which corresponds to a semi-linear continuous Hermitian form

$$\pi(u,v) := \int_\Gamma q_1^{-2}(x)\tilde{z}(x;u)\overline{\tilde{z}(x;v)}\, d\Gamma$$

$$+\sum_{i=1}^N \int_{\gamma_i} (r_i^{(1)}(x))^{-2} u_i^{(1)}(x)\overline{v_i^{(1)}(x)} d\gamma_i + \sum_{i=1}^N \int_{\gamma_i} (r_i^{(2)}(x))^{-2} u_i^{(2)}(x)\overline{v_i^{(2)}(x)} d\gamma_i,$$

and $L(u)$ a linear continuous functional defined on H. Moreover, since

$$\tilde{I}(u) \ge \sum_{i=1}^N \int_{\gamma_i} (r_i^{(1)}(x))^{-2} |u_i^{(1)}(x)|^2 d\gamma_i$$

$$+\sum_{i=1}^N \int_{\gamma_i} (r_i^{(2)}(x))^{-2} |u_i^{(2)}(x)|^2 d\gamma_i \ge c\|u\|_H \quad \forall u \in H, \ \mathrm{c{=}const},$$

we obtain, using Remark 1.1 to Theorem 1.1 from [1], that there exists one and only one element $\hat{u} = (\hat{u}_1^{(1)}, \dots, \hat{u}_N^{(1)}, \hat{u}_1^{(2)}, \dots, \hat{u}_N^{(2)}) \in H$ such that що

$$I(\hat{u}) = \inf_{u \in H} I(u).$$

Therefore, for any fixed $v \in H$ and $\tau \in \mathbb{R}^1$, the function $s(\tau) := I(\hat{u} + \tau v)$ has only one minimum point $\tau = 0$, so that

$$\frac{d}{d\tau} I(\hat{u} + \tau v) \mid_{\tau=0} = 0.$$

This yields

$$0=\frac{1}{2}\frac{d}{d\tau}I(\hat{u}+\tau v)\mid_{\tau=0}=\lim_{\tau\to 0}\frac{1}{2\tau}(I(\hat{u}+\tau v)-I(\hat{u}))$$

$$=\lim_{\tau\to 0}\frac{1}{2\tau}\int_\Gamma q^{-2}(x)\left(z(x;\hat{u}+\tau v)\overline{z(x;\hat{u}+\tau v)}-z(x;\hat{u})\overline{z(x;\hat{u})}\right)\,d\Gamma$$

$$+\lim_{\tau\to 0}\frac{1}{2\tau}\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(1)}(x))^{-2}\left[\left(\hat{u}_i^{(1)}(x)+\tau v_i^{(1)}(x)\right)\overline{\left(\hat{u}_i^{(1)}(x)+\tau v_i^{(1)}(x)\right)}\right.$$

$$\left.-\hat{u}_i^{(1)}(x)\overline{\hat{u}_i^{(1)}(x)}\right]\,d\gamma_i$$

$$+\lim_{\tau\to 0}\frac{1}{2\tau}\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(2)}(x))^{-2}\left[\left(\hat{u}_i^{(2)}(x)+\tau v_i^{(2)}(x)\right)\overline{\left(\hat{u}_i^{(2)}(x)+\tau v_i^{(2)}(x)\right)}\right.$$

$$\left.-\hat{u}_i^{(2)}(x)\overline{\hat{u}_i^{(2)}(x)}\right]\,d\gamma_i.$$

Calculate the first limit in the right-hand side of the last relationship. Taking into notice the notation for $\tilde{z}(x;v)$ and the equality $z(x;\hat{u}+\tau v)=z(x;\hat{u})+\tau\tilde{z}(x;v)$, we have

$$\lim_{\tau\to 0}\frac{1}{2\tau}\int_\Gamma q^{-2}(x)\left(z(x;\hat{u}+\tau v)\overline{z(x;\hat{u}+\tau v)}-z(x;\hat{u})\overline{z(x;\hat{u})}\right)\,d\Gamma$$

$$=\lim_{\tau\to 0}\frac{1}{2\tau}\int_\Gamma q^{-2}(x)\left[2\tau\mathrm{Re}\,(z(\cdot;\hat{u})\overline{\tilde{z}(\cdot;v)})+O(\tau^2)\right]\,d\Gamma$$

$$=\mathrm{Re}\int_\Gamma q_1^{-2}(x)z(x;\hat{u})\overline{\tilde{z}(x;v)}d\Gamma.$$

Performing similar calculations for the remaining limits we obtain

$$0=\mathrm{Re}\int_\Gamma q_1^{-2}(x)z(x;\hat{u})\overline{\tilde{z}(x;v)}d\Gamma+\sum_{i=1}^{N}\mathrm{Re}\int_{\gamma_i}(r_i^{(1)}(x))^{-2}\hat{u}_i^{(1)}(x)\overline{v_i^{(1)}(x)}d\gamma_i$$

$$+\sum_{i=1}^{N}\mathrm{Re}\int_{\gamma_i}(r_i^{(2)}(x))^{-2}\hat{u}_i^{(2)}(x)\overline{v_i^{(2)}(x)}d\gamma_i. \tag{2.91}$$

On the other side, for any fixed $v\in H$ and $\tau\in\mathbb{R}^1$, the function $s_1(\tau_1):=I(\hat{u}+i\tau_1 v)$ has the unique minimum point at $\tau_1=0$, so that

$$\frac{d}{d\tau_1}I(\hat{u}+i\tau_1 v)\mid_{\tau_1=0}=0,$$

which yields

$$0=\mathrm{Im}\int_\Gamma q_1^{-2}(x)z(x;\hat{u})\overline{\tilde{z}(x;v)}d\Gamma+\sum_{i=1}^{N}\mathrm{Im}\int_{\gamma_i}(r_i^{(1)}(x))^{-2}\hat{u}_i^{(1)}(x)\overline{v_i^{(1)}(x)}d\gamma_i$$

$$+\sum_{i=1}^{N}\operatorname{Im}\int_{\gamma_i}(r_i^{(2)}(x))^{-2}\hat{u}_i^{(2)}(x)\overline{v_i^{(2)}(x)}d\gamma_i. \tag{2.92}$$

From the latter, in line with (2.91) and (2.92), it follows

$$0=\int_{\Gamma}q_1^{-2}(x)z(x;\hat{u})\overline{\tilde{z}(x;v)}d\Gamma+\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(1)}(x))^{-2}\hat{u}_i^{(1)}(x)\overline{v_i^{(1)}(x)}d\gamma_i$$

$$+\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(2)}(x))^{-2}\hat{u}_i^{(2)}(x)\overline{v_i^{(2)}(x)}d\gamma_i. \tag{2.93}$$

Introduce function $p(x)$ as the unique solution to the problem

$$p\in H^1_{\mathrm{loc}}(\Omega',\Delta), \tag{2.94}$$

$$(\Delta+k^2)p(x)=0 \quad \text{in} \quad \Omega', \tag{2.95}$$

$$\frac{\partial p}{\partial\nu}=q_1^{-2}z(\cdot;\hat{u}) \quad \text{on} \quad \Gamma, \tag{2.96}$$

$$[p]_{\gamma_i}=0, \quad \left[\frac{\partial p}{\partial\nu}\right]_{\gamma_i}=0, \quad i=\overline{1,N}, \tag{2.97}$$

$$\frac{\partial p(x)}{\partial r}-ikp(x)=o(1/r),\ r=|x|,\ r\to\infty. \tag{2.98}$$

Transform the first term in the right-hand side of (2.91) using equalities (2.94)–(2.98) and applying the second Green formula in domains Ω_i, $i=1,N$, and $\tilde{\Omega}_R$ to functions p and $\overline{\tilde{z}(\cdot;v)}$. We have

$$0=\int_{\tilde{\Omega}_R}-(\Delta+k^2)p(x)\overline{\tilde{z}(x;v)}dx+\sum_{i=1}^{N}\int_{\Omega_i}-(\Delta+k^2)p(x)\overline{\tilde{z}(x;v)}dx$$

$$=\int_{\tilde{\Omega}_R}-\overline{(\Delta+\bar{k}^2)\tilde{z}(x;v)}p(x)\,dx-\int_{\Gamma}\overline{\tilde{z}(\cdot;v)}\frac{\partial p}{\partial\nu_A}\,d\Gamma-\Sigma_R(\tilde{z}(\cdot;v),p)$$

$$+\sum_{i=1}^{N}\int_{\Omega_i}-\overline{(\Delta+\bar{k}^2)\tilde{z}(x;v)}p(x)\,dx-\sum_{i=1}^{N}\int_{\hat{\gamma}_i}\left(\left(\overline{\frac{\partial\tilde{z}(\cdot;v)}{\partial\nu_{A^*}}}\right)_{-}p(x)-\overline{\tilde{z}_-(\cdot;v)}\frac{\partial p}{\partial\nu_A}\right)d\hat{\gamma}_i$$

$$+\sum_{i=1}^{N}\int_{\hat{\gamma}_i}\left(\left(\overline{\frac{\partial\tilde{z}(\cdot;v)}{\partial\nu_{A^*}}}\right)_{+}p-\overline{\tilde{z}_+(\cdot;v)}\frac{\partial p(x)}{\partial\nu_A}\right)d\hat{\gamma}_i$$

$$=\int_{\Omega'_R}-\overline{(\Delta+\bar{k}^2)\tilde{z}(x;v)}p(x)\,dx-\int_{\Gamma}\overline{\tilde{z}(\cdot;v)}\frac{\partial p}{\partial\nu}\,d\Gamma-\Sigma_R(\tilde{z}(\cdot;v),p)$$

$$-\sum_{i=1}^{N}\int_{\gamma_i}\left(\overline{\tilde{z}_+(\cdot;v)}-\overline{\tilde{z}_-(\cdot;v)}\right)\frac{\partial p}{\partial\nu}\,d\hat{\gamma}_i$$

$$+\sum_{i=1}^{N}\int_{\gamma_i}\left(\left(\overline{\frac{\partial\tilde z(\cdot;v)}{\partial\nu_{A^*}}}\right)_+ - \left(\overline{\frac{\partial\tilde z(\cdot;v)}{\partial\nu_{A^*}}}\right)_-\right) p\,d\gamma_i$$

$$= -\int_{\Gamma}\overline{\tilde z(\cdot;v)}q_1^{-2}(x)z(x;\hat u)\,d\Gamma - \Sigma_R(\tilde z(\cdot;v),p)$$

$$-\sum_{i=1}^{N}\int_{\gamma_i}p(x)\int_{\gamma_i}[\,K_i^{(1,1)}(\xi,x)\overline{v_i^{(1)}(\xi)}+K_i^{(2,1)}(\xi,x)\overline{v_i^{(2)}(\xi)}\,]\,d\gamma_{i_\xi}\,d\gamma_{i_x}$$

$$-\sum_{i=1}^{N}\int_{\gamma_i}\frac{\partial p(x)}{\partial\nu}\int_{\gamma_i}[\,K_i^{(1,2)}(\xi,x)\overline{v_i^{(1)}(\xi)}+K_i^{(2,2)}(\xi,x)\overline{v_i^{(2)}(\xi)}\,]\,d\gamma_{i_\xi}\,d\gamma_{i_x}.$$

Calculating the limit as $R\to\infty$ and taking into account that $\Sigma_R(\tilde z(\cdot;v),p)=o(1)$, we obtain

$$0=\int_{\Gamma}\overline{\tilde z(\cdot;v)}q_1^{-2}(x)z(x;\hat u)\,d\Gamma$$

$$\sum_{i=1}^{N}\int_{\gamma_i}p(x)\int_{\gamma_i}[\,K_i^{(1,1)}(\xi,x)\overline{v_i^{(1)}(\xi)}+K_i^{(2,1)}(\xi,x)\overline{v_i^{(2)}(\xi)}\,]\,d\gamma_{i_\xi}\,d\gamma_{i_x}$$

$$\sum_{i=1}^{N}\int_{\gamma_i}\frac{\partial p(x)}{\partial\nu}\int_{\gamma_i}[\,K_i^{(1,2)}(\xi,x)\overline{v_i^{(1)}(\xi)}+K_i^{(2,2)}(\xi,x)\overline{v_i^{(2)}(\xi)}\,]\,d\gamma_{i_\xi}\,d\gamma_{i_x}.$$

Next, by virtue of (2.93),

$$\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(1)}(x))^{-2}\hat u_i^{(1)}(x)\overline{v_i^{(1)}(x)}d\gamma_i+\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(2)}(x))^{-2}\hat u_i^{(2)}(x)\overline{v_i^{(2)}(x)}d\gamma_i$$

$$=\sum_{i=1}^{N}\int_{\gamma_i}\overline{v_i^{(1)}(x)}\left[K_i^{(1,1)}(x,\xi)p(\xi)+K_i^{(1,2)}(x,\xi)\frac{\partial p(\xi)}{\partial\nu}\right]\,d\gamma_{i_\xi}\,d\gamma_{i_x}$$

$$+\sum_{i=1}^{N}\int_{\gamma_i}\overline{v_i^{(2)}(x)}\left[K_i^{(2,1)}(x,\xi)p(\xi)+K_i^{(2,2)}(x,\xi)\frac{\partial p(\xi)}{\partial\nu}\right]\,d\gamma_{i_\xi}\,d\gamma_{i_x}.$$

Rewrite the last equality in the form

$$\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(1)}(x))^{-2}\Big[\hat u_i^{(1)}(x)-(r_i^{(1)}(x))^2\int_{\gamma_i}\Big(K_i^{(1,1)}(x,\xi)p(\xi)+$$
$$+K_i^{(1,2)}(x,\xi)\frac{\partial p(\xi)}{\partial\nu}\Big)\,d\gamma_{i_\xi}\Big]\overline{v_i^{(1)}(x)}d\gamma_{i_x} \qquad (2.99)$$

$$+\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(2)}(x))^{-2}\left[\hat{u}_i^{(2)}(x)-(r_i^{(2)}(x))^2\int_{\gamma_i}\left(K_i^{(2,1)}(x,\xi)p(\xi)+\right.\right.$$

$$\left.\left.+K_i^{(2,2)}(x,\xi)\frac{\partial p(\xi)}{\partial\nu}\right)\,d\gamma_{i_\xi}\right]\overline{v_i^{(2)}(x)}d\gamma_{i_x}=0.$$

Setting in (2.99)

$$v_i^{(1)}(x)=\hat{u}_i^{(1)}(x)-(r_i^{(1)}(x))^2\int_{\gamma_i}\left(K_i^{(1,1)}(x,\xi)p(\xi)+K_i^{(1,2)}(x,\xi)\frac{\partial p(\xi)}{\partial\nu}\right)\,d\gamma_{i_\xi},$$

$$v_i^{(2)}(x)=\hat{u}_i^{(2)}(x)-(r_i^{(2)}(x))^2\int_{\gamma_i}\left(K_i^{(2,1)}(x,\xi)p(\xi)+K_i^{(2,2)}(x,\xi)\frac{\partial p(\xi)}{\partial\nu}\right)\,d\gamma_{i_\xi},$$

$i=\overline{1,N}$, we find

$$\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(1)}(x))^{-2}\left|\hat{u}_i^{(1)}(x)-(r_i^{(1)}(x))^2\int_{\gamma_i}\left(K_i^{(1,1)}(x,\xi)p(\xi)+\right.\right.$$

$$\left.\left.+K_i^{(1,2)}(x,\xi)\frac{\partial p(\xi)}{\partial\nu}\right)\,d\gamma_{i_\xi}\right|^2 d\gamma_{i_x}$$

$$+\sum_{i=1}^{N}\int_{0\gamma_i}(r_i^{(2)}(x))^{-2}\left|\hat{u}_i^{(2)}(x)-(r_i^{(2)}(x))^2\int_{\gamma_i}\left(K_i^{(2,1)}(x,\xi)p(\xi)+\right.\right.$$

$$\left.\left.+K_i^{(2,2)}(x,\xi)\frac{\partial p(\xi)}{\partial\nu}\right)\,d\gamma_{i_\xi}\right|^2 d\gamma_{i_x}=0,$$

and consequently,

$$\hat{u}_i^{(1)}(x)=(r_i^{(1)}(x))^2\int_{\gamma_i}\left[K_i^{(1,1)}(x,\xi)p(\xi)+K_i^{(1,2)}(x,\xi)\frac{\partial p(\xi)}{\partial\nu}\right]\,d\gamma_{i_\xi},$$

$$\hat{u}_i^{(2)}(x)=(r_i^{(2)}(x))^2\int_{\gamma_i}\left[K_i^{(2,1)}(x,\xi)p(\xi)+K_i^{(2,2)}(x,\xi)\frac{\partial p(\xi)}{\partial\nu}\right]\,d\gamma_{i_\xi},\quad i=\overline{1,N}.$$

Substituting these quantities to (2.59) and (2.62), setting $z(x)=z(x;\hat{u})$, and taking into account (2.94)–(2.98), we arrive at problem (2.73)–(2.82); the unique solvability of this problem follows from the fact that functional (2.64) has the unique minimum point $\hat{u}$.

Now let us establish the validity of (2.84). Substituting expressions (2.72) to (2.64), we obtain

$$\sigma^2=I(\hat{u}_1^{(1)},\ldots,\hat{u}_N^{(1)},\hat{u}_1^{(2)},\ldots,\hat{u}_N^{(2)})=$$

$$
= \int_\Gamma q_1^{-2}(x)|z(x)|^2\, d\Gamma
$$

$$
+\sum_{i=1}^N \int_{\gamma_i} (r_i^{(1)}(x))^{-2}|\hat{u}_i^{(1)}(x)|^2 d\gamma_i + \sum_{i=1}^N \int_{\gamma_i} (r_i^{(2)}(x))^{-2}|\hat{u}_i^{(2)}(x)|^2 d\gamma_i
$$

$$
= \int_\Gamma q_1^{-2}(x)|z(x)|^2\, d\Gamma
$$

$$
+\sum_{i=1}^N \int_{\gamma_i} (r_i^{(1)}(x))^2 \Big| \int_{\gamma_i} \left[K_i^{(1,1)}(x,\xi)p(\xi) + K_i^{(1,2)}(x,\xi)\frac{\partial p(\xi)}{\partial \nu}\right] d\gamma_{i_\xi} \Big|^2 d\gamma_{i_x}
$$

$$
+\sum_{i=1}^N \int_{\gamma_i} (r_i^{(2)}(x))^2 \Big| \int_{\gamma_i} \left[K_i^{(2,1)}(x,\xi)p(\xi) + K_i^{(2,2)}(x,\xi)\frac{\partial p(\xi)}{\partial \nu}\right] d\gamma_{i_\xi} \Big|^2 d\gamma_{i_x}. \tag{2.100}
$$

Next, using relationships (2.72)−(2.82), we have

$$
0 = \int_{\tilde{\Omega}_R} -(\Delta + k^2)p(x)\overline{z(x)}\, dx + \sum_{i=1}^N \int_{\Omega_i} -(\Delta + k^2)p(x)\overline{z(x)}\, dx
$$

$$
= \int_{\tilde{\Omega}_R} -\overline{(\Delta + \bar{k}^2)z(x)}p(x)\, dx - \int_\Gamma \overline{z(x)}\frac{\partial p(x)}{\partial \nu}\, d\Gamma - \Sigma_R(z,p)
$$

$$
-\sum_{i=1}^N \int_{\hat{\gamma}_i} \left(\left(\overline{\frac{\partial z}{\partial \nu}}\right)_- p - \overline{z}_- \frac{\partial p}{\partial \nu} \right) d\hat{\gamma}_i
$$

$$
+\sum_{i=1}^N \int_{\Omega_i} -\overline{(\Delta + \bar{k}^2)z(x)}p(x)\, dx + \sum_{i=1}^N \int_{\hat{\gamma}_i} \left(\left(\overline{\frac{\partial z}{\partial \nu}}\right)_+ p - \overline{z}_+ \frac{\partial p}{\partial \nu} \right) d\hat{\gamma}_i
$$

$$
= \int_{\Omega'_R} -\overline{(\Delta + \bar{k}^2)z(x)}p(x)\, dx - \int_\Gamma \overline{z}\frac{\partial p}{\partial \nu}\, d\Gamma - \Sigma_R(z,p)
$$

$$
-\sum_{i=1}^N \int_{\gamma_i} (\overline{z}_+ - \overline{z}_-)\frac{\partial p}{\partial \nu}\, d\gamma_i + \sum_{i=1}^N \int_{\gamma_i} \left[\left(\overline{\frac{\partial z}{\partial \nu}}\right)_+ - \left(\overline{\frac{\partial z}{\partial \nu}}\right)_- \right] p\, d\gamma_i
$$

$$
= \int_{\Omega'_R} \chi_{\omega_0}(x)\overline{l_0(x)}p(x)\, dx - \int_\Gamma q_1^{-2}|z|^2\, d\Gamma - \Sigma_R(z,p)
$$

$$
-\sum_{i=1}^N \int_{\gamma_i} \frac{\partial p(x)}{\partial \nu} \int_{\gamma_i} K_i^{(1,2)}(\xi,x)(r_i^{(1)}(\xi))^2 \int_{\gamma_i} \overline{\Big[p(y) \times}
$$

$$
\overline{\times\, K_i^{(1,1)}(\xi,y) + \frac{\partial p(y)}{\partial \nu}K_i^{(1,2)}(\xi,y)\Big]}\, d\gamma_{i_y}\, d\gamma_{i_\xi}\, d\gamma_{i_x}
$$

$$
\begin{aligned}
&-\sum_{i=1}^{N}\int_{\gamma_i}\frac{\partial p(x)}{\partial\nu}\int_{\gamma_i}K_i^{(2,2)}(\xi,x)(r_i^{(2)}(\xi))^2\int_{\gamma_i}\overline{\Big[p(y)\times} \\
&\qquad\qquad\overline{\times K_i^{(2,1)}(\xi,y)+\frac{\partial p(y)}{\partial\nu}K_i^{(2,2)}(\xi,y)\Big]}\,d\gamma_{i_y}\,d\gamma_{i_\xi}\,d\gamma_{i_x}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=1}^{N}\int_{\gamma_i}p(x)\int_{\gamma_i}K_i^{(1,1)}(\xi,x)(r_i^{(1)}(\xi))^2\int_{\gamma_i}\overline{\Big[p(y)\times} \\
&\qquad\qquad\overline{\times K_i^{(1,1)}(\xi,y)+\frac{\partial p(y)}{\partial\nu}K_i^{(1,2)}(\xi,y)\Big]}\,d\gamma_{i_y}\,d\gamma_{i_\xi}\,d\gamma_{i_x}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=1}^{N}\int_{\gamma_i}p(x)\int_{\gamma_i}K_i^{(2,1)}(\xi,x)(r_i^{(2)}(\xi))^2\int_{\gamma_i}\overline{\Big[p(y)\times} \\
&\qquad\qquad\overline{\times K_i^{(2,1)}(\xi,y)+\frac{\partial p(y)}{\partial\nu}K_i^{(2,2)}(\xi,y)\Big]}\,d\gamma_{i_y}\,d\gamma_{i_\xi}\,d\gamma_{i_x}
\end{aligned}
$$

$$
\begin{aligned}
&=\int_{\omega_0}\overline{l_0(x)}p(x)\,dx-\int_\Gamma q_1^{-2}|z|^2\,d\Gamma-\Sigma_R(z,p) \\
&-\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(1)}(\xi))^2\,d\gamma_{i_\xi}\int_{\gamma_i}\overline{\Big[p(y)K_i^{(1,1)}(\xi,y)+\frac{\partial p(y)}{\partial\nu}K_i^{(1,2)}(\xi,y)\Big]}\,d\gamma_{i_y}\times \\
&\qquad\times\int_{\gamma_i}\Big[p(x)K_i^{(1,1)}(\xi,x)+\frac{\partial p(x)}{\partial\nu}K_i^{(1,2)}(\xi,x)\Big]\,d\gamma_{i_x} \\
&-\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(2)}(\xi))^2\,d\gamma_{i_\xi}\int_{\gamma_i}\overline{\Big[p(y)K_i^{(2,1)}(\xi,y)+\frac{\partial p(y)}{\partial\nu}K_i^{(2,2)}(\xi,y)\Big]}\,d\gamma_{i_y}\times \\
&\qquad\times\int_{\gamma_i}\Big[p(x)K_i^{(2,1)}(\xi,x)+\frac{\partial p(x)}{\partial\nu}K_i^{(2,2)}(\xi,x)\Big]\,d\gamma_{i_x} \\
&=\int_{\omega_0}\overline{l_0(x)}p(x)\,dx-\int_\Gamma q_1^{-2}|z|^2\,d\Gamma-\Sigma_R(z,p) \\
&-\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(1)}(x))^2\Big|\int_{\gamma_i}\Big[K_i^{(1,1)}(x,\xi)p(\xi)+K_i^{(1,2)}(x,\xi)\frac{\partial p(\xi)}{\partial\nu}\Big]\,d\gamma_{i_\xi}\Big|^2 d\gamma_{i_x} \\
&-\sum_{i=1}^{N}\int_{\gamma_i}(r_i^{(2)}(x))^2\Big|\int_{\gamma_i}\Big[K_i^{(2,1)}(x,\xi)p(\xi)+K_i^{(2,2)}(x,\xi)\frac{\partial p(\xi)}{\partial\nu}\Big]\,d\gamma_{i_\xi}\Big|^2 d\gamma_{i_x}.
\end{aligned}
\tag{2.101}
$$

Equality (2.84) follows now from two relationships (2.100) and (2.101) and the fact that $\Sigma_R(z,p)=o(1)$ when $R\to\infty$. □

An alternative representation for the minimax estimate in terms of the solution to a system of integro-differential equations is given in the next theorem. This solution is independent of the specific form of functional (2.12).

Theorem 2.5. *The minimax estimate of (2.12) has the form*

$$\widehat{\widehat{l(\varphi)}} = l(\hat{\varphi}) = \int_{\omega_0} \overline{l_0(x)}\hat{\varphi}(x)\,dx, \tag{2.102}$$

where function $\hat{\varphi}(x)$ is determined from the solution to the problem (2.103)–(2.112):

$$\hat{p} \in L^2(\Sigma, H^1_{\mathrm{loc}}(\Omega', \Delta))^{13}, \tag{2.103}$$

$$(\Delta + \bar{k}^2)\hat{p}(x,\omega) = 0 \quad in \quad \Omega', \tag{2.104}$$

$$\frac{\partial \hat{p}(\cdot,\omega)}{\partial \nu} = 0 \quad on \quad \Gamma, \tag{2.105}$$

$$[\hat{p}(x,\omega))]_{\gamma_i} = -\int_{\gamma_i} \overline{K_i^{(1,2)}(\xi,x)}\left[(r_i^{(1)}(\xi))^2 y_i^{(1)}(\xi,\omega) - \hat{v}_i^{(1)}(\xi,\omega)\right] d\gamma_{i_\xi} - \\ - \int_{\gamma_i} \overline{K_i^{(2,2)}(\xi,x)}\left[(r_i^{(2)}(\xi))^2 y_i^{(2)}(\xi,\omega) - \hat{v}_i^{(2)}(\xi,\omega)\right] d\gamma_{i_\xi},$$

$$\left[\frac{\partial \hat{p}(x,\omega))}{\partial \nu}\right]_{\gamma_i} = \int_{\gamma_i} \overline{K_i^{(1,1)}(\xi,x)}\left[(r_i^{(1)}(\xi))^2 y_i^{(1)}(\xi,\omega) - \hat{v}_i^{(1)}(\xi,\omega)\right] d\gamma_{i_\xi} + \\ + \int_{\gamma_i} \overline{K_i^{(2,1)}(\xi,x)}\left[(r_i^{(2)}(\xi))^2 y_i^{(2)}(\xi,\omega) - \hat{v}_i^{(2)}(\xi,\omega)\right] d\gamma_{i_\xi}$$

$$on \quad \gamma_i, \quad i = \overline{1,N}, \tag{2.106}$$

$$\frac{\partial \hat{p}(x,\omega)}{\partial r} + i\bar{k}\hat{p}(x,\omega) = o(1/r),\ r = |x|,\ r \to \infty, \tag{2.107}$$

$$\hat{\varphi} \in L^2(\Sigma, H^1_{\mathrm{loc}}(\Omega', \Delta)), \tag{2.108}$$

$$(\Delta + k^2)\hat{\varphi}(x,\omega) = 0 \quad in \quad \Omega', \tag{2.109}$$

$$\frac{\partial \hat{\varphi}(\cdot,\omega)}{\partial \nu} = q_1^{-2}\hat{p}(\cdot,\omega) + h_0 \quad on \quad \Gamma, \tag{2.110}$$

$$[\hat{\varphi}(\cdot,\omega)]_{\gamma_i} = 0, \quad \left[\frac{\partial \hat{\varphi}(\cdot,\omega)}{\partial \nu}\right]_{\gamma_i} = 0, \quad i = \overline{1,N}, \tag{2.111}$$

$$\frac{\partial \hat{\varphi}(x,\omega)}{\partial r} - ik\hat{\varphi}(x,\omega) = o(1/r),\ r = |x|,\ r \to \infty, \tag{2.112}$$

[13] Here $L^2(\Sigma, H^1_{\mathrm{loc}}(\Omega', \Delta))$ denotes a class of functions the belong to $L^2(\Sigma, H^1(\Omega'_R, \Delta))$ for any $R > 0$.

where

$$\hat{v}_i^{(1)}(\xi,\omega) = (r_i^{(1)}(\xi))^2 \int_{\gamma_i} \left[K_i^{(1,1)}(\xi,\eta)\hat{\varphi}(\eta,\omega) + K_i^{(1,2)}(\xi,\eta)\frac{\partial\hat{\varphi}(\eta,\omega)}{\partial\nu} \right] d\gamma_{i_\eta}, \quad (2.113)$$

$$\hat{v}_i^{(2)}(\xi,\omega) = (r_i^{(2)}(\xi))^2 \int_{\gamma_i} \left[K_i^{(2,1)}(\xi,\eta)\hat{\varphi}(\eta,\omega) + K_i^{(2,2)}(\xi,\eta)\frac{\partial\hat{\varphi}(\eta,\omega)}{\partial\nu} \right] d\gamma_{i_\eta}, \quad (2.114)$$

and the right-hand sides in (2.106) are considered for every realization of random functions $y_i^{(1)}(\xi) = y_i^{(1)}(\xi,\omega)$ *and* $y_i^{(2)}(\xi) = y_i^{(2)}(\xi,\omega)$ *which belong with probability* 1 *to the space* $L^2(\gamma_i)$, $i = \overline{1,N}$. *Problem (2.103)–(2.112) is uniquely solvable.*

Proof. The proof of this theorem is similar to the proof of Theorems 1.2 and 2.4. $\square$

Remark 6. *Function* $\hat{\varphi}(x,\omega)$ *which is determined from the solution to problem (2.103)–(2.11 can be taken as a good estimate of the unknown solution* $\varphi(x)$ *to the initial Neumann problem (2.1)–(2.3) (see Remark 1 on p. 31).*

Set

$$\tilde{K}_i^{(1,1)}(x,\eta) = \int_{\gamma_i} \Big[K_i^{(1,1)}(\xi,\eta)\overline{K_i^{(1,2)}(\xi,x)}(r_i^{(1)}(\xi))^2 + K_i^{(2,1)}(\xi,\eta)\overline{K_i^{(2,2)}(\xi,x)}(r_i^{(2)}(\xi))^2 \Big] d\gamma_{i_\eta}, \quad (2.115)$$

$$\tilde{K}_i^{(1,2)}(x,\eta) = \int_{\gamma_i} \Big[K_i^{(1,2)}(\xi,\eta)\overline{K_i^{(1,2)}(\xi,x)}(r_i^{(1)}(\xi))^2 + K_i^{(2,2)}(\xi,\eta)\overline{K_i^{(2,2)}(\xi,x)}(r_i^{(2)}(\xi))^2 \Big] d\gamma_{i_\eta}, \quad (2.116)$$

$$\tilde{K}_i^{(2,1)}(x,\eta) = -\int_{\gamma_i} \Big[K_i^{(1,1)}(\xi,\eta)\overline{K_i^{(1,1)}(\xi,x)}(r_i^{(1)}(\xi))^2 + K_i^{(2,1)}(\xi,\eta)\overline{K_i^{(2,1)}(\xi,x)}(r_i^{(2)}(\xi))^2 \Big] d\gamma_{i_\eta}, \quad (2.117)$$

$$\tilde{K}_i^{(2,2)}(x,\eta) = -\int_{\gamma_i} \Big[K_i^{(1,2)}(\xi,\eta)\overline{K_i^{(1,1)}(\xi,x)}(r_i^{(1)}(\xi))^2 + K_i^{(2,2)}(\xi,\eta)\overline{K_i^{(2,1)}(\xi,x)}(r_i^{(2)}(\xi))^2 \Big] d\gamma_{i_\eta}, \quad (2.118)$$

Then Theorem 2.4 can be formulated as follows.

Theorem 2.6. *The minimax estimate of the value of functional $l(\varphi)$ has the form*

$$\widehat{\widehat{l(\varphi)}} = \sum_{i=1}^{N} \int_{\gamma_i} \left(\overline{\hat{u}_i^{(1)}(x)} y_i^{(1)}(x) + \overline{\hat{u}_i^{(2)}(x)} y_i^{(2)}(x) \right) d\gamma_i + \hat{c}, \tag{2.119}$$

where $\hat{c} = \int_\Gamma \bar{z} g_0 \, d\Gamma$,

$$\hat{u}_i^{(1)}(x) = (r_i^{(1)}(x))^2 \int_{\gamma_i} \left[K_i^{(1,1)}(x,\xi) p(\xi) + K_i^{(1,2)}(x,\xi) \frac{\partial p(\xi)}{\partial \nu} \right] d\gamma_{i_\xi}, \tag{2.120}$$

$$\hat{u}_i^{(2)}(x) = (r_i^{(2)}(x))^2 \int_{\gamma_i} \left[K_i^{(2,1)}(x,\xi) p(\xi) + K_i^{(2,2)}(x,\xi) \frac{\partial p(\xi)}{\partial \nu} \right] d\gamma_{i_\xi}, \quad i = \overline{1,N},$$

and functions z and p are determined from the uniquely solvable problem

$$z \in H^1_{\mathrm{loc}}(\Omega', \Delta), \tag{2.121}$$

$$-(\Delta + \bar{k}^2) z(x) = \chi_{\omega_0}(x) l_0(x) \text{ in } \Omega', \tag{2.122}$$

$$\frac{\partial z}{\partial \nu} = 0 \text{ on } \Gamma, \tag{2.123}$$

$$[z(x)]_{\gamma_i} = \int_{\gamma_i} \left[\tilde{K}_i^{(1,1)}(x,\eta) p(\eta) + \tilde{K}_i^{(1,2)}(x,\eta) \frac{\partial p(\eta)}{\partial \nu} \right] d\gamma_{i_\eta}, \text{ on } \gamma_i, \ i = \overline{1,N}, \tag{2.124}$$

$$\left[\frac{\partial z(x)}{\partial \nu} \right]_{\gamma_i} = \int_{\gamma_i} \left[\tilde{K}_i^{(2,1)}(x,\eta) p(\eta) + \tilde{K}_i^{(2,2)}(x,\eta) \frac{\partial p(\eta)}{\partial \nu} \right] d\gamma_{i_\eta}, \text{ on } \gamma_i, \ i = \overline{1,N}, \tag{2.125}$$

$$\frac{\partial z}{\partial r} + i\bar{k} z = o(1/r), \ r = |x|, \ r \to \infty, \tag{2.126}$$

$$p \in H^1_{\mathrm{loc}}(\Omega', \Delta), \tag{2.127}$$

$$(\Delta + k^2) p(x) = 0 \text{ in } \Omega', \tag{2.128}$$

$$\frac{\partial p}{\partial \nu} = Q^{-1} z_0 \text{ on } \Gamma, \tag{2.129}$$

$$[p(x)]_{\gamma_i} = 0, \quad \left[\frac{\partial p(x)}{\partial \nu} \right]_{\gamma_i} = 0, \text{ on } \gamma_i, \ i = \overline{1,N}, \tag{2.130}$$

$$\frac{\partial p}{\partial r} - ikp = o(1/r), \ r = |x|, \ r \to \infty. \tag{2.131}$$

2.4 Integral equation systems whose solutions are used to express minimax estimates

In the previous section, we have obtained the integro-differential equations whose solutions are used to express minimax estimates. In this section, we use the developed potential theory in Sobolev spaces and reduce these integro-differential equations to integral equations over an unclosed surface which is a union of the boundary of domain Ω and surfaces γ_i, $i=\overline{1,N}$, on which observations are made. This reduction allows one to decrease the dimensionality of the problem of finding minimax estimates.

We define first, in addition to the single- and double-layer potentials introduced in the previous sections, the corresponding boundary integral operators S_k, K_k, K_k', and T_k:

$$(S_k\varphi)(x) := 2\int_\Gamma \Phi_k(x,y)\varphi(y)\,d\Gamma_y,\ \ x\in\Gamma, \tag{2.132}$$

$$(K_k\varphi)(x) := 2\int_\Gamma \frac{\partial\Phi_k(x,y)}{\partial\nu_y}\varphi(y)\,d\Gamma_y,\ \ x\in\Gamma, \tag{2.133}$$

$$(K_k'\psi)(x) := 2\int_\Gamma \frac{\partial\Phi_k(x,y)}{\partial\nu_x}\psi(y)\,d\Gamma_y,\ \ x\in\Gamma, \tag{2.134}$$

$$(T_k\psi)(x) := 2\frac{\partial}{\partial\nu_x}\int_\Gamma \frac{\partial\Phi_k(x,y)}{\partial\nu_y}\psi(y)\,d\Gamma_y,\ \ x\in\Gamma; \tag{2.135}$$

in the three-dimensional case, their kernels are determined by the formulas

$$\frac{\partial\Phi_k(x,y)}{\partial\nu_y} = \Phi_k(x,y)\frac{(x-y,\nu_y)}{|x-y|^2}\left(1-ik|x-y|\right), \tag{2.136}$$

$$\frac{\partial\Phi_k(x,y)}{\partial\nu_x} = \Phi_k(x,y)\frac{(x-y,\nu_x)}{|x-y|^2}\left(ik|x-y|-1\right), \tag{2.137}$$

$$\begin{aligned}\frac{\partial^2\Phi_k(x,y)}{\partial\nu_x\partial\nu_y} = \Phi_k(x,y)\Bigg[&\frac{(\nu_x,\nu_y)}{|x-y|^2}\left(1-ik|x-y|\right)\\ &+\frac{(x-y,\nu_x)(x-y,\nu_y)}{|x-y|^4}\left(-3+3ik|x-y|+k^2|x-y|^3\right)\Bigg];\end{aligned} \tag{2.138}$$

in the two-dimensional case,

$$\frac{\partial\Phi_k(x,y)}{\partial\nu_y} = \frac{ik(x-y,\nu_y)}{4|x-y|}H_1^{(1)}(k|x-y|), \tag{2.139}$$

$$\frac{\partial\Phi_k(x,y)}{\partial\nu_x} = -\frac{ik(x-y,\nu_x)}{4|x-y|}H_1^{(1)}(k|x-y|), \tag{2.140}$$

$$
\frac{\partial^2 \Phi_k(x,y)}{\partial \nu_x \partial \nu_y} = \frac{ikH_1^{(1)}(k|x-y|)}{4|x-y|^3}\Big[(\nu_x,\nu_y)|x-y|^2 \\
-(x-y,\nu_x)(x-y,\nu_y)(|x-y|+1)\Big] + \frac{ik^2 H_0^{(1)}(k|x-y|)}{4|x-y|^2}(x-y,\nu_x)(x-y,\nu_y), \tag{2.141}
$$

where $H_1^1(z)$ denotes the order-one Hankel function of the first kind and $(\cdot,\cdot)$ the inner product in $\mathbb{R}^n$, $n = 2, 3$.

Note that, for example, in the three-dimensional case, the kernels of integral operators (2.132)−(2.134) have a weak singularity and the integral in the right-hand side of (2.135) is understood as a Cauchy singular integral.

Let us formulate the properties of the operators introduced above that are essential for the reduction of problem (2.121)−(2.131) to a system of surface integral equations.

If Γ is a C^2-surface, then the following operators are continuous at $|s| \le 1/2$:

$$
\begin{aligned}
S_k &: H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma), \\
K_k &: H^{1/2+s}(\Gamma) \to H^{3/2+s}(\Gamma), \\
K'_k &: H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma), \\
T_k &: H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma)
\end{aligned} \tag{2.142}
$$

(similar statements are valid for the operators $S_{-\bar{k}}$, $K_{-\bar{k}}$, $K'_{-\bar{k}}$, and $T_{-\bar{k}}$).

Also, operators K_k and $K'_{-\bar{k}}$ acting, respectively, from $H^{1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ and from $H^{-1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ are compact according to (2.142) and the following equalities hold:

$$
\begin{aligned}
(K_k\varphi,\psi)_{L^2(\Gamma)} &= (\varphi, K'_{-\bar{k}}\psi)_{L^2(\Gamma)} \ \forall \varphi \in H^{1/2}(\Gamma), \psi \in H^{-1/2}(\Gamma), \\
(S_k\varphi,\psi)_{L^2(\Gamma)} &= (\varphi, S_{-\bar{k}}\psi)_{L^2(\Gamma)} \quad \forall \varphi \in H^{-1/2}(\Gamma), \psi \in H^{-1/2}(\Gamma), \\
(T_k\varphi,\psi)_{L^2(\Gamma)} &= (\varphi, T_{-\bar{k}}\psi)_{L^2(\Gamma)} \ \ \forall \varphi \in H^{1/2}(\Gamma), \psi \in H^{1/2}(\Gamma),
\end{aligned} \tag{2.143}
$$

where

$$
(f,g)_{L^2(\Gamma)} = \int_\Gamma f\bar{g}\, d\Gamma =: g(f)
$$

for every $f \in H^{1/2}(\Gamma)$ and $g \in H^{-1/2}(\Gamma)$.

Denote by $\left(\mathcal{V}_\Gamma^k\psi\right)^+(x)$ and $\left(\mathcal{W}_\Gamma^k\psi\right)^+(x)$ the restriction on the domain $\mathbb{R}^3 \setminus \bar{\Omega}$ of the single- and double-layer potentials (2.19) and (2.20) with a density $\psi \in H^s(\Gamma)$, $s \in \mathbb{R}$.

The traces on $\Gamma = \partial\Omega$ of these functions and their derivatives satisfy the relations [51], pp. 224, 225:

$$\left(\mathcal{V}_\Gamma^k \psi\right)^+ = \frac{1}{2} S_k \psi \quad \text{in} \quad H^{s+1}(\Gamma), \tag{2.144}$$

$$\left(\mathcal{W}_\Gamma^k \psi\right)^+ = \frac{1}{2}(\psi + K_k \psi) \quad \text{in} \quad H^s(\Gamma), \tag{2.145}$$

$$\frac{\partial}{\partial \nu}\left(\mathcal{V}_\Gamma^k \psi\right)^+ = -\frac{1}{2}(\psi - K'_k \psi) \quad \text{in} \quad H^s(\Gamma). \tag{2.146}$$

$$\frac{\partial}{\partial \nu}\left(\mathcal{W}_\Gamma^k \psi\right)^+ = \frac{1}{2} T_k \psi \quad \text{in} \quad H^{s-1}(\Gamma). \tag{2.147}$$

Similar relations are valid if we replace k by $-\bar{k}$.

Denote by $D(\Omega)$ a countable set of positive values of wave number k with a limiting point at infinity such that the homogeneous internal Nuemann problem

$$\Delta v + k^2 v = 0 \text{ in } \Omega, \tag{2.148}$$

$$v = 0 \text{on} \Gamma \tag{2.149}$$

has nontrivial solutions. Then [52]

$$N(I - K'_k) = \left\{ \left.\frac{\partial v}{\partial \nu}\right|_\Gamma : \Delta v + k^2 v = 0 \text{ in } \Omega,\ v = 0 \text{ on } \Gamma \right\}, \tag{2.150}$$

where $N(I - K'_k)$ denotes the null-space of $I - K'_k$.

It is known that the solution $v_1(x)$ of the problem

$$\Delta v_1(x) + k^2 v_1(x) = 0 \text{ in } \mathbb{R}^3 \setminus \bar{\Omega}, \tag{2.151}$$

$$\frac{\partial v_1}{\partial \nu} = f_1 \text{ on } \Gamma, \tag{2.152}$$

$$\frac{\partial v_1}{\partial r} - ikv_1 = o(1/r),\ r = |x|,\ r \to \infty \text{ if } \operatorname{Im} k \geq 0, \tag{2.153}$$

can be represented as

$$v_1(x) = \mathcal{W}_\Gamma^k \varphi_1(x) - \mathcal{V}_\Gamma^k f_1(x),\ x \in \mathbb{R}^3 \setminus \bar{\Omega}, \tag{2.154}$$

where the function $\varphi_1 := v_1|_\Gamma$ which is the trace of this solution on Γ can be determined directly or as a solution to the integral equation

$$\varphi_1(x) - K_k \varphi_1(x) = -S_k f_1(x),\ x \in \Gamma, \tag{2.155}$$

when $k \notin D(\Omega)$; generally (that is, for any $k \neq 0$, $\operatorname{Im} k \geq 0$) it can be found from the equation

$$\varphi_1(x) - K_k\varphi_1(x) - i\eta T_k\varphi_1(x) = -S_k f_1(x) - i\eta(f_1(x) + K'_k f_1(x)),\ x \in \Gamma, \quad (2.156)$$

in which a number $\eta \in \mathbb{R}$, $\eta \neq 0$, is chosen so that

$$\eta \operatorname{Re} k > 0. \quad (2.157)$$

The existence of solutions to these integral equations at any $f_1 \in H^{-1/2}(\Gamma)$ follows from the unique solvability of BVP (2.151)–(2.153). The solution to (2.155) is unique because

$$\dim N(I - K_k) = \dim N(I - K'_k) = \dim N(I - K'_{-\bar{k}}) = 0 \quad (2.158)$$

due to (2.150) and the Fredholm alternative; the uniqueness for (2.156) is a consequence of the fact that the operator $I - K_k - i\eta T_k : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ defined by the left-hand side of (2.156) is an isomorphism [51].

A reasoning similar to that in [51]–[53] enables us to prove that the solution $v_2(x)$ to the problem

$$\Delta v_2(x) + \bar{k}^2 v_2(x) = 0 \text{ in } \mathbb{R}^3 \setminus \bar{\Omega}, \quad (2.159)$$

$$\frac{\partial v_2}{\partial \nu} = f_2 \text{ on } \Gamma, \quad (2.160)$$

$$\frac{\partial v_2}{\partial r} + i\bar{k} v_2 = o(1/r),\ r = |x|,\ r \to \infty, \quad (2.161)$$

can be represented as

$$v_2(x) = \mathcal{W}_\Gamma^{-\bar{k}}\varphi_2(x) - \mathcal{V}_\Gamma^{-\bar{k}} f_2(x),\ x \in \mathbb{R}^3 \setminus \bar{\Omega}, \quad (2.162)$$

where the function $\varphi_2 := v_2|_\Gamma$ which is a trace of this solution on Γ can be determined directly or as a solution to the integral equation

$$\varphi_2(x) - K_{-\bar{k}}\varphi_2(x) = -S_{-\bar{k}} f_2(x),\ x \in \Gamma, \quad (2.163)$$

for $k \notin D(\Omega)$; generally, this function can be determined from the integral equation (for any $k \neq 0$, $\operatorname{Im} k \geq 0$)

$$\varphi_2(x) - K_{-\bar{k}}\varphi_2(x) + i\eta T_{-\bar{k}}\varphi_2(x)$$
$$= -S_{-\bar{k}} f_2(x) + i\eta(f_2(x) + K'_{-\bar{k}} f_2(x)),\ x \in \Gamma, \quad (2.164)$$

where η satisfies conditon (2.157).

Indeed, the solution $v_2 \in H^1(\mathbb{R}^3 \setminus \bar{\Omega})$ of the Helmholtz equation (2.159) that satisfies radiation conditions (2.161) and the property $\frac{\partial v_2}{\partial \nu} \in L^1(\Gamma)$ admits the integral representation

$$v_2(x) = \int_\Gamma \frac{\partial \Phi_{-\bar{k}}(x,y)}{\partial \nu_y} v_2(y)\, d\Gamma_y - \int_\Gamma \Phi_{-\bar{k}}(x,y) \frac{\partial v_2(y)}{\partial \nu}\, d\Gamma_y, \quad x \in \mathbb{R}^3 \setminus \bar{\Omega}; \tag{2.165}$$

replacing in this formula $\frac{\partial v_2}{\partial \nu}\big|_\Gamma$ by f_2, we obtain equality (2.162) for the solution $v_2(x)$ to problem (2.159)–(2.161).

Calculating the traces on Γ for both sides of (2.162) and using relationships (2.144) and (2.145) with k replaced by $-\bar{k}$, obtain a boundary integral equation for φ_2

$$\varphi_2 - K_{-\bar{k}}\varphi_2 = -S_{-\bar{k}} f_2 \quad \text{on} \quad \Gamma; \tag{2.166}$$

taking into account (2.146)–(2.147), we can obtain another integral equation for φ_2

$$T_{-\bar{k}}\varphi_2 = f_2 + K'_{-\bar{k}} f_2 = 0 \quad \text{on} \quad \Gamma. \tag{2.167}$$

Multiplying both sides of (2.167) by a number η and adding to (2.167), we find that φ_2 satisfies integral equation (2.164).

The uniqueness of solution of this integral equation follows from the fact that operator $K_{-\bar{k}}$ is adjoint to K'_k and operator $T_{-\bar{k}}$ is adjoint to T_k; therefore, $I - K_{-\bar{k}} + i\eta T_{-\bar{k}}$ is adjoint to $I - K'_k - i\eta T_k$. Also,

$$\operatorname{Im}(I - K_{-\bar{k}} + i\eta T_{-\bar{k}}) = \operatorname{Ker}(I - K'_k - i\eta T_k)^\perp.$$

In [52] it is proved (see Theorem 3.34) that under condition (2.157) $\operatorname{Ker}(I - K'_k - i\eta T_k) = \emptyset$. Thus, if (2.157) holds, then integral equation (2.164) is uniquely solvable.

In order to reduce problem (2.121)–(2.131) in unbounded domain $\mathbb{R}^3 \setminus \bar{\Omega}$ to a system of boundary integral equations let us apply the results formulated above in Subsection 2.5.

Introduce functions $\varrho_i^{(1)}(\hat{u}_i^{(1)}, \hat{u}_i^{(2)}) \in H_{00}^{1/2}(\gamma_i)$ and $\varrho_i^{(2)}(\hat{u}_i^{(1)}, \hat{u}_i^{(2)}) \in L^2(\gamma_i)$ defined on γ_i, $i = \overline{1, N}$:

$$\begin{aligned} \varrho_i^{(1)}(\hat{u}_i^{(1)}, \hat{u}_i^{(2)}) &= \varrho_i^{(1)}(\cdot; \hat{u}_i^{(1)}, \hat{u}_i^{(2)}) \\ &:= \int_{\gamma_i} \left[\overline{K_i^{(1,2)}(\xi, \cdot)} \hat{u}_i^{(1)}(\xi) + \overline{K_i^{(2,2)}(\xi, \cdot)} \hat{u}_i^{(2)}(\xi) \right] d\gamma_{i_\xi}, \end{aligned} \tag{2.168}$$

$$\varrho_i^{(2)}(\hat{u}_i^{(1)}, \hat{u}_i^{(2)}) = \varrho_i^{(1)}(\cdot; \hat{u}_i^{(1)}, \hat{u}_i^{(2)})$$

$$:= -\int_{\gamma_i} \left[\overline{K_i^{(1,1)}(\xi, \cdot)} \hat{u}_i^{(1)}(\xi) + \overline{K_i^{(2,1)}(\xi, \cdot)} \hat{u}_i^{(2)}(\xi) \right] d\gamma_{i_\xi}, \quad (2.169)$$

Introduce also a function $z_{in} = z_{in}(\cdot; \hat{u}) \in H^1_{\text{loc}}(\mathbb{R}^n \setminus (\cup_{i=1}^N \bar{\gamma}_i), \Delta)$ which solves the BVP

$$-(\Delta + \bar{k}^2) z_{in} = \chi_{\omega_0}(x) l_0 \text{ in } \mathbb{R}^n \setminus (\cup_{i=1}^N \bar{\gamma}_i), \quad (2.170)$$

$$[z_{in}]_{\gamma_i} = \varrho_i^{(1)}(\hat{u}_i^{(1)}, \hat{u}_i^{(2)}), \quad \left[\frac{\partial z_{in}}{\partial \nu} \right]_{\gamma_i} = \rho_i^{(2)}(\hat{u}_i^{(1)}, \hat{u}_i^{(2)}) \text{ on } \gamma_i, \ i = \overline{1, N}, \quad (2.171)$$

$$\frac{\partial z_{in}}{\partial r} + i\bar{k} z_{in} = o(1/r), \ r = |x|, \ r \to \infty; \quad (2.172)$$

in line with Theorem 2.2, this function, in the domain $\mathbb{R}^n \setminus (\cup_{i=1}^N \bar{\gamma}_i)$, is determined according to

$$z_{in} = \Sigma_{i=1}^N \left(\mathcal{V}_{\gamma_i}^{-\bar{k}} \varrho_i^{(2)}(\hat{u}_i^{(1)}, \hat{u}_i^{(2)}) - \mathcal{W}_{\gamma_i}^{-\bar{k}} \varrho_i^{(1)}(\hat{u}_i^{(1)}, \hat{u}_i^{(2)}) \right) + N_{-\bar{k}} l_0$$

$$= 2\Sigma_{i=1}^N \left(\int_{\gamma_i} \Phi_{-\bar{k}}(\cdot, y) \varrho_i^{(2)}(y; \hat{u}_i^{(1)}, \hat{u}_i^{(2)}) \, d\gamma_{i_y} - \int_{\gamma_i} \frac{\partial \Phi_{-\bar{k}}(\cdot, y)}{\partial \nu_{i_y}} \varrho_i^{(1)}(y; \hat{u}_i^{(1)}, \hat{u}_i^{(2)}) \, d\gamma_{i_y} \right)$$

$$+ \int_{\omega_0} \Phi_{-\bar{k}}(\cdot, y) l_0(y) \, dy. \quad (2.173)$$

Then the solution to problem (2.121)–(2.126) can be represented in Ω' as

$$z = z_s + z_{in}, \quad (2.174)$$

where the function $z_s = z_s(\cdot; \hat{u})$,

$$\hat{u} = (\hat{u}_1^{(1)}, \ldots, \hat{u}_N^{(1)}, \hat{u}_1^{(2)}, \ldots, \hat{u}_N^{(2)}) \in \left(L^2(\gamma_1) \times \ldots \times L^2(\gamma_N) \right)^2,$$

is determined from the solution to the following problem

$$z_s \in H^1_{\text{loc}}(\Omega', \Delta), \quad (2.175)$$

$$(\Delta + \bar{k}^2) z_s = 0 \text{ in } \Omega', \quad (2.176)$$

$$\frac{\partial z_s}{\partial \nu} = -\frac{\partial z_{in}}{\partial \nu} \text{ on } \Gamma, \quad (2.177)$$

$$\frac{\partial z_s}{\partial r} + i\bar{k} z_s = o(1/r), \ r = |x|, \ r \to \infty, \quad (2.178)$$

where

$$\frac{\partial z_{in}}{\partial \nu} = \Sigma_{i=1}^{N}\left(\frac{\partial \mathcal{V}_{\gamma_i}^{-\bar{k}}\varrho_i^{(2)}(\hat{u}_i^{(1)},\hat{u}_i^{(2)})}{\partial \nu} - \frac{\partial \mathcal{W}_{\gamma_i}^{-\bar{k}}\varrho_i^{(1)}(\hat{u}_i^{(1)},\hat{u}_i^{(2)})}{\partial \nu}\right) + \frac{\partial N_{-\bar{k}}l_0}{\partial \nu}$$

$$= 2\Sigma_{i=1}^{N}\left(\int_{\gamma_i}\frac{\partial \Phi_{-\bar{k}}(\cdot,y)}{\partial \nu}\varrho_i^{(2)}(y;\hat{u}_i^{(1)},\hat{u}_i^{(2)})d\gamma_{i_y} - \int_{\gamma_i}\frac{\partial^2 \Phi_{-\bar{k}}(\cdot,y)}{\partial \nu \partial \nu_y}\varrho_i^{(1)}(y;\hat{u}_i^{(1)},\hat{u}_i^{(2)})d\gamma_{i_y}\right) + \int_{\omega_0}\frac{\partial \Phi_{-\bar{k}}(\cdot,y)}{\partial \nu}l_0(y)\,dy. \quad (2.179)$$

According to (2.162) and (2.164), where $v_2 = z_s$, $f_2 = -\frac{\partial z_{in}}{\partial \nu}\big|_\Gamma$, and $\varphi_2 = z_s|_\Gamma$, the trace of z_s on Γ denoted by

$$\psi := z_s|_\Gamma = z|_\Gamma - z_{in}|_\Gamma \quad (2.180)$$

(see (2.174)) satisfies the integral equation

$$\psi - K_{-\bar{k}}\psi + i\eta T_{-\bar{k}}\psi = S_{-\bar{k}}\left.\frac{\partial z_{in}}{\partial \nu}\right|_\Gamma - i\eta\left(\left.\frac{\partial z_{in}}{\partial \nu}\right|_\Gamma + K'_{-\bar{k}}\left.\frac{\partial z_{in}}{\partial \nu}\right|_\Gamma\right). \quad (2.181)$$

Introduce also the notations

$$\chi := p|_\Gamma, \quad \varphi_i^{(1)} := p|_{\gamma_i}, \quad \varphi_i^{(2)} := \left.\frac{\partial p}{\partial \nu}\right|_{\gamma_i}, \quad i = \overline{1,N}. \quad (2.182)$$

Then, by virtue of (2.81) and (2.174) $\frac{\partial p}{\partial \nu} = q_1^{-2}z = q_1^{-2}(\psi + z_{in})$ on Γ.

Taking into account relationships (2.154), (2.156) in which we set $f_1 = q_1^{-2}z = q_1^{-2}(\psi + z_{in})$, $\varphi_1 = \chi$, and $v_1 = p$, we obtain an integral representation for the solution p of BVP (2.80)−(2.83) in the domain $\mathbb{R}^3 \setminus \bar{\Omega}$

$$p = \mathcal{W}_\Gamma^k\chi - \mathcal{V}_\Gamma^k q_1^{-2}(\psi + z_{in}\,|_\Gamma), i = \overline{1,N}, \quad (2.183)$$

where $\chi = p|_\Gamma$ satisfies on Γ a boundary integral equation

$$(I - K_k - i\eta T_k)\chi = -(S_k + i\eta(I + K'_k))q_1^{-2}(\psi + z_{in}\,|_\Gamma), \quad (2.184)$$

From (2.183) it follows that

$$\left.\frac{\partial p}{\partial \nu}\right|_{\gamma_i} = \left.\frac{\partial \mathcal{W}_\Gamma^k\chi}{\partial \nu}\right|_{\gamma_i} - \left.\frac{\partial \mathcal{V}_\Gamma^k q_1^{-2}(\psi + z_{in}\,|_\Gamma)}{\partial \nu}\right|_{\gamma_i}, \ i = \overline{1,N}. \quad (2.185)$$

and

$$p|_{\gamma_i} = \mathcal{W}_\Gamma^k\chi\,|_{\gamma_i} - \mathcal{V}_\Gamma^k q_1^{-2}(\psi + z_{in}\,|_\Gamma)\,|_{\gamma_i}, i = \overline{1,N}, \quad (2.186)$$

Since $\varphi_i^{(1)} = p|_{\gamma_i}$ and $\varphi_i^{(2)} = \frac{\partial p}{\partial \nu}\Big|_{\gamma_i}$, $i = \overline{1, N}$, the latter equalities combined with (2.181) and (2.184) mean that functions ψ, χ, $\varphi_i^{(1)}$, $\varphi_i^{(2)}$ and $\hat{u}_i^{(1)}$, $\hat{u}_i^{(2)}$ defined by (2.180), (2.182), and (2.72) solve the following integral equation system

$$(I - K_{-\bar{k}} + i\eta T_{-\bar{k}})\psi = (S_{-\bar{k}} - i\eta(I + K'_{-\bar{k}})) \left.\frac{\partial z_{in}}{\partial \nu}\right|_{\Gamma}, \tag{2.187}$$

$$(I - K_k - i\eta T_k)\chi = -(S_k + i\eta(I + K'_k))q_1^{-2}(\psi + z_{in}\,|_{\Gamma}), \tag{2.188}$$

$$\varphi_i^{(1)} = \mathcal{W}_\Gamma^k \chi\,|_{\gamma_i} - \mathcal{V}_\Gamma^k q_1^{-2}(\psi + z_{in}\,|_{\Gamma})\,|_{\gamma_i}, i = \overline{1, N}, \tag{2.189}$$

$$\varphi_i^{(2)} = \left.\frac{\partial \mathcal{W}_\Gamma^k \chi}{\partial \nu}\right|_{\gamma_i} - \left.\frac{\partial \mathcal{V}_\Gamma^k q_1^{-2}(\psi + z_{in}\,|_{\Gamma})}{\partial \nu}\right|_{\gamma_i}, \quad i = \overline{1, N}, \tag{2.190}$$

$$\hat{u}_i^{(1)} = (r_i^{(1)})^2 \int_{\gamma_i} \left[K_i^{(1,1)}(\cdot, \xi)\varphi_i^{(1)}(\xi) + K_i^{(1,2)}(\cdot, \xi)\varphi_i^{(2)}(\xi)\right] d\gamma_{i_\xi}, \tag{2.191}$$

$$\hat{u}_i^{(2)} = (r_i^{(2)})^2 \int_{\gamma_i} \left[K_i^{(2,1)}(\cdot, \xi)\varphi_i^{(1)}(\xi) + K_i^{(2,2)}(\cdot, \xi)\varphi_i^{(2)}(\xi)\right] d\gamma_{i_\xi}, \quad i = \overline{1, N}. \tag{2.192}$$

Resolve this system with respect to functions $\hat{u}_i^{(1)}$ and $\hat{u}_i^{(2)}$. Replace in (2.168) and (2.169) $\hat{u}_i^{(1)}$ and $\hat{u}_i^{(2)}$ by their expressions (2.191) and (2.192) to obtain $\varrho_i^{(1)}(\hat{u}_i^{(1)}, \hat{u}_i^{(2)}) = \rho_i^{(1)}(\varphi_i^{(1)}, \varphi_i^{(2)})$ and $\varrho_i^{(2)}(\hat{u}_i^{(1)}, \hat{u}_i^{(2)}) = \rho_i^{(2)}(\varphi_i^{(1)}, \varphi_i^{(2)})$ where

$$\rho_i^{(1)}(\varphi_i^{(1)}, \varphi_i^{(2)}) := \int_{\gamma_i} \left[\tilde{K}_i^{(1,1)}(\cdot, \eta)\varphi_i^{(1)}(\eta) + \tilde{K}_i^{(1,2)}(\cdot, \eta)\varphi_i^{(2)}(\eta)\right] d\gamma_{i_\eta}, \tag{2.193}$$

$$\rho_i^{(2)}(\varphi_i^{(1)}, \varphi_i^{(2)}) := \int_{\gamma_i} \left[\tilde{K}_i^{(2,1)}(\cdot, \eta)\varphi_i^{(1)}(\eta) + \tilde{K}_i^{(2,2)}(\cdot, \eta)\varphi_i^{(2)}(\eta)\right] d\gamma_{i_\eta}, \tag{2.194}$$

and $\tilde{K}_i^{(i,j)}(\cdot, \cdot)$, $i, j = 1, 2$, are determined according to (2.115)–(2.118). Next, replacing in (2.173) functions $\varrho_i^{(1)}(\hat{u}_i^{(1)}, \hat{u}_i^{(2)})$ by $\rho_i^{(1)}(\varphi_i^{(1)}, \varphi_i^{(2)})$ and $\varrho_i^{(2)}(\hat{u}_i^{(1)}, \hat{u}_i^{(2)})$ by $\rho_i^{(2)}(\varphi_i^{(1)}, \varphi_i^{(2)})$ we have

$$z_{in} = \Sigma_{i=1}^N \left(\mathcal{V}_{\gamma_i}^{-\bar{k}} \rho_i^{(2)}(\varphi_i^{(1)}, \varphi_i^{(2)}) - \mathcal{W}_{\gamma_i}^{-\bar{k}} \rho_i^{(1)}(\varphi_i^{(1)}, \varphi_i^{(2)})\right) + N_{-\bar{k}} l_0. \tag{2.195}$$

Substituting this expression into (2.187)–(2.190) we conclude that ψ, χ, $\varphi_i^{(1)}$, and $\varphi_i^{(2)}$ satisfy the integral equation system

$$\begin{aligned}&(I - K_{-\bar{k}} + i\eta T_{-\bar{k}})\psi = (S_{-\bar{k}} - i\eta(I + K'_{-\bar{k}}))\\ &\left(\Sigma_{i=1}^N \left(\left.\frac{\partial \mathcal{V}_{\gamma_i}^{-\bar{k}} \rho_i^{(2)}(\varphi_i^{(1)}, \varphi_i^{(2)})}{\partial \nu}\right|_{\Gamma} - \left.\frac{\partial \mathcal{W}_{\gamma_i}^{-\bar{k}} \rho_i^{(1)}(\varphi_i^{(1)}, \varphi_i^{(2)})}{\partial \nu}\right|_{\Gamma}\right) + \left.\frac{\partial N_{-\bar{k}} l_0}{\partial \nu}\right|_{\Gamma}\right),\end{aligned} \tag{2.196}$$

$$(I-K_k-i\eta T_k)\chi=-(S_k+i\eta(I+K_k'))$$
$$q_1^{-2}\left(\psi+\Sigma_{i=1}^N\left(\mathcal{V}_{\gamma_i}^{-\bar k}\rho_i^{(2)}(\varphi_i^{(1)},\varphi_i^{(2)})\Big|_\Gamma-\mathcal{W}_{\gamma_i}^{-\bar k}\rho_i^{(1)}(\varphi_i^{(1)},\varphi_i^{(2)})\Big|_\Gamma\right)+N_{-\bar k}l_0|_\Gamma\right), \quad (2.197)$$

$$\varphi_i^{(1)}=\mathcal{W}_\Gamma^k\chi\big|_{\gamma_i}-\mathcal{V}_\Gamma^k q_1^{-2}\Big(\psi+\sum_{i=1}^N\Big(\mathcal{V}_{\gamma_i}^{-\bar k}\rho_i^{(2)}(\varphi_i^{(1)},\varphi_i^{(2)})\Big|_\Gamma$$
$$-\mathcal{W}_{\gamma_i}^{-\bar k}\rho_i^{(1)}(\varphi_i^{(1)},\varphi_i^{(2)})\Big|_\Gamma\Big)+N_{-\bar k}l_0|_\Gamma\Big)\Big|_{\gamma_i},\ i=\overline{1,N}, \quad (2.198)$$
$$\varphi_i^{(2)}=\frac{\partial\mathcal{W}_\Gamma^k\chi}{\partial\nu}\bigg|_{\gamma_i}$$
$$-\left.\frac{\partial\mathcal{V}_\Gamma^k q_1^{-2}\left(\psi+\Sigma_{i=1}^N\left(\mathcal{V}_{\gamma_i}^{-\bar k}\rho_i^{(2)}(\varphi_i^{(1)},\varphi_i^{(2)})\Big|_\Gamma-\mathcal{W}_{\gamma_i}^{-\bar k}\rho_i^{(1)}(\varphi_i^{(1)},\varphi_i^{(2)})\Big|_\Gamma\right)+N_{-\bar k}l_0\,|_\Gamma\right)}{\partial\nu}\right|_{\gamma_i},$$
$$i=\overline{1,N}. \quad (2.199)$$

Summing up the above reasoning and taking into consideration Theorems 2.4 and 2.6 we arrive at the following result.

Theorem 2.7. *The minimax estimate of $l(\varphi)$ has the form*

$$\widehat{\widehat{l(\varphi)}}=\sum_{i=1}^N\int_{\gamma_i}\left(\overline{\hat u_i^{(1)}(x)}y_i^{(1)}(x)+\overline{\hat u_i^{(2)}(x)}y_i^{(2)}(x)\right)d\gamma_i+\hat c, \quad (2.200)$$

where

$$\hat c=\int_\Gamma\overline{\left[\psi+\sum_{i=1}^N\left(\mathcal{V}_i^{-\bar k}\rho_i^{(2)}(\varphi_i^{(1)},\varphi_i^{(2)})\Big|_\Gamma-\mathcal{W}_i^{-\bar k}\rho_i^{(1)}(\varphi_i^{(1)},\varphi_i^{(2)})\Big|_\Gamma\right)+N_{-\bar k}l_0\,|_\Gamma\right]}g_0\,d\Gamma, \quad (2.201)$$

$$\hat u_i^{(1)}(x)=(r_i^{(1)}(x))^2\int_{\gamma_i}\left[K_i^{(1,1)}(x,\xi)\varphi_i^{(1)}(\xi)+K_i^{(1,2)}(x,\xi)\varphi_i^{(2)}(\xi)\right]d\gamma_{i_\xi}, \quad (2.202)$$
$$\hat u_i^{(2)}(x)=(r_i^{(2)}(x))^2\int_{\gamma_i}\left[K_i^{(2,1)}(x,\xi)\varphi_i^{(1)}(\xi)+K_i^{(2,2)}(x,\xi)\varphi_i^{(1)}(\xi)\right]d\gamma_{i_\xi},\quad i=\overline{1,N}.$$

The auxiliary function

$$\psi:=z|_\Gamma-\Sigma_{i=1}^N\left(\mathcal{V}_{\gamma_i}^{-\bar k}\rho_i^{(2)}(\varphi_i^{(1)},\varphi_i^{(2)})\Big|_\Gamma-\mathcal{W}_{\gamma_i}^{-\bar k}\rho_i^{(1)}(\varphi_i^{(1)},\varphi_i^{(2)})\Big|_\Gamma\right)-N_{-\bar k}l_0|_\Gamma$$

defined on Γ *and functions* $\chi := p|_\Gamma$, $\varphi_i^{(1)} := p|_{\gamma_i}$, *and* $\varphi_i^{(2)} := \frac{\partial p}{\partial \nu_{\gamma_i}}$, $i = \overline{1,N}$, *are determined from the solution to the integral equation system* (2.196)–(2.199), *in which* η *is an arbitrary real number such that* $\eta \mathrm{Re}\, k > 0$. *This system is uniquely solvable for all values of wave numbers* k, $\mathrm{Im}\, k \geq 0$.

The estimation error $\sigma = l(p)^{1/2}$, *where*

$$p = \mathcal{W}_\Gamma^k \chi - \mathcal{V}_\Gamma^k q_1^{-2}\Big(\psi + \Sigma_{i=1}^N \Big(\mathcal{V}_{\gamma_i}^{-\bar{k}} \rho_i^{(2)}(\varphi_i^{(1)}, \varphi_i^{(2)})\Big|_\Gamma \\ - \mathcal{W}_{\gamma_i}^{-\bar{k}} \rho_i^{(1)}(\varphi_i^{(1)}, \varphi_i^{(2)})\Big|_\Gamma\Big) + N_{-\bar{k}} l_0 \,|_\Gamma\Big) \quad (2.203)$$

in the domain $\mathbb{R}^3 \setminus \bar{\Omega}$.

Proof. It is necessary to prove only the unique solvability of system (2.196)–(2.199) which follows from the unique solvability of the system of integro-differential equations (2.72)–(2.83).

Indeed, let the integral equation system (2.196)–(2.199) has another solution $\tilde{\psi}$, $\tilde{\chi}$, $\tilde{\varphi}_i^{(1)}$, $\tilde{\varphi}_i^{(2)}$, $i = \overline{1,N}$. Introduce functions $\tilde{\hat{u}}_i^{(1)}(x)$, $\tilde{\hat{u}}_i^{(2)}(x)$, $\tilde{p}$, and $\tilde{z}_{in}$ by formulas (2.191), (2.192), (2.203), and (2.195) in which ψ, χ, $\varphi_i^{(1)}$, and $\varphi_i^{(2)}$, $i = \overline{1,N}$ are replaced by $\tilde{\psi}$, $\tilde{\chi}$, $\tilde{\varphi}_i^{(1)}$, and $\tilde{\varphi}_i^{(2)}$, $i = \overline{1,N}$, and functions $\tilde{z}_s$ and $\tilde{z}$ by $\tilde{z}_s = \mathcal{W}_\Gamma^{-\bar{k}} \tilde{\psi} - \mathcal{V}_\Gamma^{-\bar{k}} \tilde{z}_{in}|_\Gamma$ and $\tilde{z} = \tilde{z}_s + \tilde{z}_{in}$. Then from (2.198)–(2.199) it follows that $\tilde{\varphi}_i^{(1)} = \tilde{p}|_{\gamma_i}$ and $\tilde{\varphi}_i^{(2)} = \frac{\partial \tilde{p}}{\partial \nu}\Big|_{\gamma_i}$, $i = \overline{1,N}$. Theorem 2.2 and the equalities $\varrho_i^{(1)}(\tilde{\hat{u}}_i^{(1)}, \tilde{\hat{u}}_i^{(2)}) = \rho_i^{(1)}(\tilde{\varphi}_i^{(1)}, \tilde{\varphi}_i^{(2)})$, $\varrho_i^{(2)}(\tilde{\hat{u}}_i^{(1)}, \tilde{\hat{u}}_i^{(2)}) = \rho_i^{(2)}(\tilde{\varphi}_i^{(1)}, \tilde{\varphi}_i^{(2)})$ imply that $\tilde{z}$ and $\tilde{p}$ will also satisfy integro-differential equation system (2.72)–(2.83) which is uniquely solvable. The latter statement and the fact that operators $I - K_k - i\eta T_k$, $I - K_{-\bar{k}} + i\eta T_{-\bar{k}} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are isomorphic mappings yield $\tilde{\varphi}_i^{(1)} = \varphi_i^{(1)}$, $\tilde{\varphi}_i^{(2)} = \varphi_i^{(2)}$, $i = \overline{1,N}$, and $\tilde{\psi} = \psi$, $\tilde{\chi} = \chi$. $\square$

Using Theorem 2.5, the notations

$$D_{-\bar{k}} := \sum_{i=1}^N \left(\mathcal{V}_i^{-\bar{k}} d_i^{(2)} - \mathcal{W}_i^{-\bar{k}} d_i^{(1)}\right),$$

$$d_i^{(1)} := -\int_{\gamma_i} \overline{K_i^{(1,2)}(\xi,\cdot)} (r_i^{(1)}(\xi))^2 y_i^{(1)}(\xi,\omega)\, d\gamma_i - \int_{\gamma_i} \overline{K_i^{(2,2)}(\xi,\cdot)} (r_i^{(2)}(\xi))^2 y_i^{(2)}(\xi,\omega)\, d\gamma_i,$$

$$d_i^{(2)} := \int_{\gamma_i} \overline{K_i^{(1,1)}(\xi,\cdot)} (r_i^{(1)}(\xi))^2 y_i^{(1)}(\xi,\omega)\, d\gamma_i + \int_{\gamma_i} \overline{K_s^{(2,2)}(\xi,\cdot)} (r_i^{(2)}(\xi))^2 y_i^{(2)}(\xi,\omega)\, d\gamma_i,$$

$\tilde{\psi} := \hat{p}_s|_\Gamma - D_{-\bar{k}}\,|_\Gamma$, $\tilde{\chi} := \hat{\varphi}|_\Gamma$, $\tilde{\varphi}_i^{(1)} := \hat{\varphi}|_{\gamma_i}$, $\tilde{\varphi}_i^{(2)} := \frac{\partial \hat{\varphi}(\xi)}{\partial \nu}\Big|_{\gamma_i}$, $i = \overline{1,N}$, and the reasoning that led to the proof of Theorem 2.7, we can prove the following

Theorem 2.8. *The minimax estimate of $l(\varphi)$ has the form*

$$\widehat{\widehat{l(\varphi)}} = l(\hat{\varphi}), \tag{2.204}$$

where

$$\hat{\varphi} = \mathcal{W}_\Gamma^k \tilde{\chi} - \mathcal{V}_\Gamma^k \Big[q_1^{-2} \Big(\tilde{\psi} + \sum_{i=1}^N \Big(\mathcal{V}_{\gamma_i}^{-\bar{k}} \rho_i^{(2)}(\tilde{\varphi}_i^{(1)}, \tilde{\varphi}_i^{(2)}) \Big|_\Gamma - \mathcal{W}_{\gamma_i}^{-\bar{k}} \rho_i^{(1)}(\tilde{\varphi}_i^{(1)}, \tilde{\varphi}_i^{(2)}) \Big|_\Gamma \Big) + D_{-\bar{k}} \, |_\Gamma \Big) + h_0 \Big], \tag{2.205}$$

in the domain $\mathbb{R}^3 \setminus \bar{\Omega}$ and functions $\tilde{\psi}$, $\tilde{\chi}$, $\tilde{\varphi}_i^{(1)}$, and $\tilde{\varphi}_i^{(2)}$, $i = \overline{1, N}$, are determined from the solution to the uniquely solvable integral equation system

$$(I - K_{-\bar{k}} + i\eta T_{-\bar{k}}) \tilde{\psi} = (S_{-\bar{k}} - i\eta(I + K'_{-\bar{k}})) \left(\Sigma_{i=1}^N \left(\frac{\partial \mathcal{V}_{\gamma_i}^{-\bar{k}} \rho_i^{(2)}(\tilde{\varphi}_i^{(1)}, \tilde{\varphi}_i^{(2)})}{\partial \nu} \Bigg|_\Gamma - \frac{\partial \mathcal{W}_{\gamma_i}^{-\bar{k}} \rho_i^{(1)}(\tilde{\varphi}_i^{(1)}, \tilde{\varphi}_i^{(2)})}{\partial \nu} \Bigg|_\Gamma \right) + \frac{\partial D_{-\bar{k}}}{\partial \nu} \Bigg|_\Gamma \right), \tag{2.206}$$

$$(I - K_k - i\eta T_k) \tilde{\chi} = -(S_k + i\eta(I + K'_k)) \left[q_1^{-2} \left(\tilde{\psi} + \sum_{i=1}^N \Big(\mathcal{V}_i^{-\bar{k}} \rho_i^{(2)}(\tilde{\varphi}_i^{(1)}, \tilde{\varphi}_i^{(2)}) \,|_\Gamma - \mathcal{W}_i^{-\bar{k}} \rho_i^{(1)}(\tilde{\varphi}_i^{(1)}, \tilde{\varphi}_i^{(2)}) \,|_\Gamma \Big) + D_{-\bar{k}} \,|_\Gamma \right) + h_0 \right], \tag{2.207}$$

$$\tilde{\varphi}_i^{(1)} = \mathcal{W}_k \tilde{\chi} \,|_{\gamma_i} - \mathcal{V}_k \Big[q_1^{-2} \Big(\tilde{\psi} + \sum_{i=1}^N \Big(\mathcal{V}_i^{-\bar{k}} \rho_i^{(2)}(\tilde{\varphi}_i^{(1)}, \tilde{\varphi}_i^{(2)}) \,|_\Gamma - \mathcal{W}_i^{-\bar{k}} \rho_i^{(1)}(\tilde{\varphi}_i^{(1)}, \tilde{\varphi}_i^{(2)}) \,|_\Gamma \Big) + D_{-\bar{k}} \,|_\Gamma \Big) + h_0 \Big] \Big|_{\gamma_i}, \quad i = \overline{1, N}, \tag{2.208}$$

$$\tilde{\varphi}_i^{(2)} = \frac{\partial \mathcal{W}_k \tilde{\chi}}{\partial \nu} \Bigg|_{\gamma_i} - \frac{\partial \mathcal{V}_k \Big[q_1^{-2} \Big(\tilde{\psi} + \sum_{i=1}^N \Big(\mathcal{V}_i^{-\bar{k}} \rho_i^{(2)}(\tilde{\varphi}_i^{(1)}, \tilde{\varphi}_i^{(2)}) \,|_\Gamma - \mathcal{W}_i^{-\bar{k}} \rho_i^{(1)}(\tilde{\varphi}_i^{(1)}, \tilde{\varphi}_i^{(2)}) \,|_\Gamma \Big) + D_{-\bar{k}} \,|_\Gamma \Big) + h_0 \Big]}{\partial \nu} \Bigg|_{\gamma_i}, \quad i = \overline{1, N}, \tag{2.209}$$

where η is an arbitrary real number such that $\eta \mathrm{Re}\, k > 0$ and functions $\rho_i^{(1)}(\tilde{\varphi}_i^{(1)}, \tilde{\varphi}_i^{(2)})$ and $\rho_i^{(2)}(\tilde{\varphi}_i^{(1)}, \tilde{\varphi}_i^{(2)})$ are determined by (2.193) and (2.194) in which $\varphi_i^{(1)}$ and $\varphi_i^{(2)}$ should be replaced, respectively, by $\tilde{\varphi}_i^{(1)}$ and $\tilde{\varphi}_i^{(2)}$.

Note that integral equation systems (2.196)–(2.199) and (2.206)–(2.209) are singular.

If $k \notin D(\Omega)$, then repeating the reasoning used in the proof of Theorems 2.7 and 2.8 where equalities of the form (2.156) and (2.164) are replaced, respectively, by those of the form (2.155) and (2.163) we see that the minimax estimate of $l(\varphi)$ may be found from (2.200)–(2.202) or (2.204), (2.205), where the functions ψ, χ, $\varphi_i^{(1)}$, $\varphi_i^{(2)}$, $i = \overline{1, N}$, and $\tilde{\psi}$, $\tilde{\chi}$, $\tilde{\varphi}_i^{(1)}$, $\tilde{\varphi}_i^{(2)}$, $i = \overline{1, N}$, are determined from the solutions of the weakly singular integral equation systems (2.196)–(2.199) and (2.206)–(2.209) with $\eta = 0$.

Remark 7. *The assumption that surfaces γ_i, $i = \overline{1, N}$, are pairwise non-overlapping is not essential. Slightly changing the proof, one can extend all results of this chapter to the case when surfaces γ_i intersect on a finite system of contours.*

Remark 8. *The method proposed in this chapter enables one to solve the problem of the minimax estimation of the value of a functional defined on $\Phi(x,t) := Re\left[e^{-i\omega t}\varphi(x)\right]$ of the form*

$$L(\Phi) := \int_{t_0}^{T}\int_{\omega_0} l_0(x,t)\Phi(x,t)\,dx\,dt$$

from the observations

$$y_i^{(1)}(x,t) = \int_{\gamma_i} K_i^{(1,1)}(x,\xi)\Phi(\xi,t)\,d\gamma_{i_\xi} + \int_{\gamma_i} K_i^{(1,2)}(x,\xi)\frac{\partial\Phi(\xi,t)}{\partial\nu}\,d\gamma_{i_\xi} + \eta_i^{(1)}(x,t),$$
$$y_i^{(2)}(x,t) = \int_{\gamma_i} K_i^{(2,1)}(x,y)\Phi(\xi,t)\,d\gamma_{i_\xi} + \int_{\gamma_i} K_i^{(2,2)}(x,y)\frac{\partial\Phi(\xi,t)}{\partial\nu}\,d\gamma_{i_\xi} + \eta_i^{(2)}(x,t),$$

$i = \overline{1, N}$, in a time interval from $t = t_0$ to $t = T$. Here we assume that for $f \in G_0$, $\omega > 0$, $l_0 \in L^2(\omega_0 \times (t_0, T))$ is a given function, and $\eta_i^{(1)}(x,t)$ and $\eta_i^{(2)}(x,t)$ are observations errors which are realizations of random fields defined on $\gamma_i \times (t_0, T)$ that are continuous in the mean-square sense and have zero expectation and unknown second moments $\mathbf{E}|\eta_i^{(1)}(x,t)|^2$ and $\mathbf{E}|\eta_i^{(2)}(x,t)|^2$ satisfying the inequality

$$\sum_{i=1}^{N}\int_{t_0}^{T}\int_{\gamma_i}\mathbf{E}|\eta_i^{(1)}(x,t)|^2\left(r_i^{(1)}(x,t)\right)^2 d\gamma_i\,dt$$
$$+\sum_{i=1}^{N}\int_{t_0}^{T}\int_{\gamma_i}\mathbf{E}|\eta_i^{(2)}(x,t)|^2\left(r_i^{(2)}(x,t)\right)^2 d\gamma_i\,dt \le 1,$$

where $r_i^{(1)}(x,t), r_i^{(2)}(x,t)$ are given functions continuous on $\gamma_i \times (t_0, T)$, $i = \overline{1, N}$, that do not vanish on these sets.

PART 3

Minimax estimation of the solutions to the boundary value problems from point observations

In the previous chapters we looked for estimates of unknown solutions (and the right-hand sides of equations entering the statements of the corresponding problems) from the observations of these solutions distributed on a system of subdomains or surfaces. In this chapter, we consider similar problems in the case of point observations and propose constructive minimax estimation methods.

Let x'_k, $k = \overline{1, N}$ and x_k, $k = \overline{1, m}$ be given systems of points belonging to domain $\mathbb{R}^3 \setminus \bar{\Omega}$. The problem is as follows: to estimate the expression

$$l(\varphi) = \sum_{i=1}^{m} \bar{a}_i \varphi(x_i), \tag{3.1}$$

from the observations of the form

$$y_k = \varphi(x'_k) + \eta_k, \;\; k = \overline{1, N}, \tag{3.2}$$

that correspond to the system state φ described by problem (2.1)–(2.4) in the class of estimates

$$\widehat{l(\varphi)} = \sum_{k=1}^{N} \bar{u}_i y_k + c, \tag{3.3}$$

linear with respect to observations (3.2) under the following assumptions: $h \in G_0$ and $\eta := (\eta_1, \dots, \eta_N) \in G_1$, where the set G_0 is given by formula (2.8), η_i are errors of observations (3.2) that are realizations of random quantities $\eta_i = \eta_i(\omega)$, G_1 is the set of random vectors $\tilde{\eta} = (\tilde{\eta}_1, \dots, \tilde{\eta}_N)$ with zero expectations and finite second moments satisfying the condition

$$\sum_{i=1}^{N} r_i^2 \mathbf{E} |\tilde{\eta}_i|^2 \leq 1, \tag{3.4}$$

and $u_i \in \mathbb{C}$, $i = \overline{1, N}$,, $a_i \in \mathbb{C}$, $i = \overline{1, m}$, $q_i \in \mathbb{R}$, $i = \overline{1, N}$, and $r_i \neq 0$ are given numbers.

Set $u := (u_1, \dots, u_N) \in \mathbb{R}^N$.

Definition. *The estimate*

$$\widehat{\widehat{l(\varphi)}} = \sum_{i=1}^{N} \overline{\hat{u}_i} y_i + \hat{c},$$

in which numbers $\hat{u}_i$ and $\hat{c}$ are determined from the condition

$$\sup_{\tilde{h}\in G_0,\,\tilde{\eta}\in G_1} \mathbf{E}|l(\tilde{\varphi}) - \widehat{l(\tilde{\varphi})}|^2 \to \inf_{u\in\mathbb{R}^N,\,c\in\mathbb{C}}, \tag{3.5}$$

where

$$\widehat{l(\tilde{\varphi})} = \sum_{i=1}^{N} \overline{u_i}\tilde{y}_i + c, \tag{3.6}$$

$$\tilde{y}_i = \tilde{\varphi}(x_i) + \tilde{\eta}_i, \quad i = \overline{1,N}, \tag{3.7}$$

and $\tilde{\varphi}(x)$ is the solution to the Neumann BVP at $h = \tilde{h}$, will be called the minimax estimate of expression (3.1).

The quantity

$$\sigma := \{\sup_{\tilde{h}\in G_0,\,\tilde{\eta}\in G_1} \mathbf{E}|l(\tilde{\varphi}) - \widehat{\widehat{l(\tilde{\varphi})}}|^2\}^{1/2} \tag{3.8}$$

will be called the error of the minimax estimation of $l(\varphi)$.

Based on the proof similar to that of Lemma 2.1 (in fact, much simpler) we can show that the following statement is valid in the case of point observations.

Lemma 3.1. *Finding the minimax estimate of functional $l(\varphi)$ is equivalent to the problem of optimal control of the system described by BVP*

$$z(\cdot;u) \in \mathcal{D}'(\mathbb{R}^3 \setminus \bar{\Omega}),$$

$$\Delta z(x;u) + \bar{k}^2 z(x;u) = \sum_{i=1}^{m} a_i\delta(x - x_i) - \sum_{k=1}^{N} u_k\delta(x - x_k') \text{ in } \mathbb{R}^3 \setminus \bar{\Omega},$$

$$\frac{\partial z(\cdot;u)}{\partial \nu} = 0 \text{ on } \Gamma,$$

$$\frac{\partial z(\cdot;u)}{\partial r} + i\bar{k}z(\cdot;u) = o(1/r),\ r = |x|,\ r \to \infty.$$

with the cost function

$$I(u) = \int_\Gamma q_1^{-2}(x)z^2(x;u)\,d\Gamma + \sum_{i=1}^{N} r_i^{-2}|u_i|^2 \to \min_{u\in\mathbb{R}^N}. \tag{3.9}$$

Starting from this lemma and proceeding with the reasoning that led from Lemma 2.1 to Theorems 2.4 and 2.5, we arrive at the following result

Theorem 3.1. *The minimax estimate of $l(\varphi)$ has the form*

$$\widehat{\widehat{l(\varphi)}} = \sum_{i=1}^{N} \overline{\hat{u}_i} y_i + \hat{c} = l(\hat{\varphi}), \tag{3.10}$$

where

$$\hat{u}_k = r_k^2 p(x_k'), \quad k = \overline{1,N}, \quad \hat{c} = \int_{\Gamma} \overline{z} h_0 \, d\Gamma, \tag{3.11}$$

the functions $z, p, \in \mathcal{D}'(\mathbb{R}^3 \setminus \bar{\Omega})$ and $\hat{\varphi} = \hat{\varphi}(\cdot, \omega) \in \mathcal{D}'(\mathbb{R}^3 \setminus \bar{\Omega})$ are determined, respectively, from the solution to the following problems:

$$-(\Delta + \bar{k}^2) z(x) = \sum_{i=1}^{m} a_i \delta(x - x_i) - \sum_{k=1}^{N} r_k^2 p(x_k') \delta(x - x_k') \text{ in } \Omega, \tag{3.12}$$

$$\frac{\partial z}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3.13}$$

$$\frac{\partial z}{\partial r} + i\bar{k} z = o(1/r), \; r = |x|, \; r \to \infty, \tag{3.14}$$

$$\Delta p(x) + k^2 p(x) = 0 \text{ in } \Omega, \tag{3.15}$$

$$\frac{\partial p}{\partial \nu} = q_1^{-2} z, \text{ on } \Gamma, \tag{3.16}$$

$$\frac{\partial p}{\partial r} - ikp = o(1/r), \; r = |x|, \; r \to \infty. \tag{3.17}$$

and

$$-(\Delta + \bar{k}^2) \hat{p}(x) = \sum_{i=1}^{N} r_i^2 [y(x_k') - \varphi(x_k')] \delta(x - x_k') \text{ in } \Omega, \tag{3.18}$$

$$\frac{\partial \hat{p}}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3.19}$$

$$\frac{\partial \hat{p}}{\partial r} + i\bar{k}\hat{p} = o(1/r), \; r = |x|, \; r \to \infty, \tag{3.20}$$

$$\Delta \hat{\varphi}(x) + k^2 \hat{\varphi}(x) = 0 \text{ in } \Omega, \tag{3.21}$$

$$\frac{\partial \hat{\varphi}}{\partial \nu} = q_1^{-2} \hat{p} + h_0 \text{ on } \Gamma, \tag{3.22}$$

$$\frac{\partial \hat{\varphi}}{\partial r} - ik\hat{\varphi} = o(1/r), \; r = |x|, \; r \to \infty. \tag{3.23}$$

Problem (3.12)–(3.23) is uniquely solvable. The following estimate is valid for the error σ of the minimax estimation of $l(\varphi)$

$$\sigma = [l(p)]^{1/2} = \left(\sum_{i=1}^{m} \bar{a}_i p(x_i) \right)^{1/2}. \tag{3.24}$$

In conclusion, we formulate the statements similar to Theorems 2.7 and 2.8 that enable one to reduce, in line with the algorithm applied in the proof of Theorem 3.1, the determination of minimax estimates to a problem of less dimensionality.

Theorem 3.2. *The minimax estimate of* $l(\varphi)$ *has the form*

$$\widehat{\widehat{l(\varphi)}} = \sum_{k=1}^{N} \bar{\hat{u}}_k y_k + \hat{c}, \tag{3.25}$$

where

$$\hat{u}_k = r_k^2 p(x_k'),\ \ k = \overline{1, N}, \tag{3.26}$$

$$\hat{c} = \overline{\int_\Gamma \left[\psi + \left(\sum_{j=1}^{m} a_j \Phi_{-\bar{k}}(\cdot - x_j) - \sum_{l=1}^{N} q_l^2 p(x_l') \Phi_{-\bar{k}}(\cdot - x_l') \right)\Big|_\Gamma \right] g_0\, d\Gamma}, \tag{3.27}$$

and functions

$$\psi := z|_\Gamma - \left(\sum_{j=1}^{m} a_j \Phi_{-\bar{k}}(\cdot - x_j) - \sum_{l=1}^{N} q_l^2 p(x_l') \Phi_{-\bar{k}}(\cdot - x_l') \right)\Bigg|_\Gamma ,$$

$\chi := p|_\Gamma$, *and numbers* $p(x_l')$, $l = \overline{1, N}$, *are determined from the solution of the following equation system:*

$$(I - K'_{-\bar{k}} + i\eta T_{-\bar{k}})\psi = (-S_{-\bar{k}} + i\eta(I + K'_{-\bar{k}}))$$
$$\left(-\sum_{j=1}^{m} a_j \left.\frac{\partial \Phi_{-\bar{k}}(\cdot - x_j)}{\partial \nu}\right|_\Gamma + \sum_{l=1}^{N} r_l^2 p(x_l') \left.\frac{\partial \Phi_{-\bar{k}}(\cdot - x_l')}{\partial \nu}\right|_\Gamma \right), \tag{3.28}$$

$$(I - K_k - i\eta T_k)\chi \tag{3.29}$$
$$= (-S_k - i\eta(I + K_k'))q_1^{-2} \left(\psi + \sum_{j=1}^{m} a_j\ \Phi_{-\bar{k}}(\cdot - x_j)|_\Gamma - \sum_{l=1}^{N} r_l^2 p(x_l')\ \Phi_{-\bar{k}}(\cdot - x_l')|_\Gamma \right),$$

$$p(x_i') = \mathcal{W}_k \chi(x_r') \tag{3.30}$$
$$-\mathcal{V}_k q_1^{-2} \left(\psi + \sum_{j=1}^{m} a_j\ \Phi_{-\bar{k}}(\cdot - x_j)|_\Gamma - \sum_{l=1}^{N} r_l^2 p(x_l')\ \Phi_{-\bar{k}}(\cdot - x_l')|_\Gamma \right)\Bigg|_{x_i'} ,\ \ i = \overline{1, N}.$$

in which η *is an arbitrary real number such that* $\eta \operatorname{Re} k > 0$. *This system is uniquely solvable for all values of wave numbers* k, $\operatorname{Im} k \geq 0$.

Equation (3.30) may be rewritten in the form

$$[1-\alpha_{ii}]p(x_i') + \sum_{l=1,\, l\neq i}^{N} \alpha_{li} p(x_l') = \mathcal{W}_k \chi(x_i') - \mathcal{V}_k q_1^{-2} \psi(x_i') + \beta_i, \;\; i = \overline{1, N}, \tag{3.31}$$

where

$$\alpha_{ls} = \frac{1}{2\pi} r_l^2 \int_\Gamma \frac{e^{ik|x_s'-y|}}{|x_s'-y|} q_1^{-2}(y) \frac{e^{-i\bar{k}|y-x_l'|}}{|y-x_l'|}\, d\Gamma_y, \quad l, s = \overline{1, N},$$

$$\beta_s = \frac{1}{2\pi} \sum_{j=1}^{m} a_j \int_\Gamma \frac{e^{ik|x_s'-y|}}{|x_s'-y|} q_1^{-2}(y) \frac{e^{-i\bar{k}|y-x_j|}}{|y-x_j|}\, d\Gamma_y, \quad s = \overline{1, N}.$$

Theorem 3.3. *The minimax estimate of $l(\varphi)$ has the form*

$$\widehat{\widehat{l(\varphi)}} = l(\hat{\varphi}) = \sum_{i=1}^{m} \bar{a}_i \hat{\varphi}(x_i), \tag{3.32}$$

where

$$\hat{\varphi} \;=\; \mathcal{W}_\Gamma^k \tilde{\chi} - \mathcal{V}_\Gamma^k \Big[q_1^{-2} \Big(\tilde{\psi} + \sum_{l=1}^{N} r_l^2 (y_l - \hat{\varphi}(x_l'))\, \Phi_{-\bar{k}}(\cdot - x_l')|_\Gamma + h_0 \Big] \text{ in } \mathbb{R}^3 \setminus \bar{\Omega} \tag{3.33}$$

and functions

$$\tilde{\psi} := \hat{p}|_\Gamma - \sum_{l=1}^{N} r_l^2 (y_l - \hat{\varphi}(x_l'))\, \Phi_{-\bar{k}}(\cdot - x_l')|_\Gamma\,,$$

$\tilde{\chi} := \hat{\varphi}|_\Gamma$, *and numbers* $\hat{\varphi}(x_l')$, $l = \overline{1, N}$, *are determined from the solution of the following equation system:*

$$(I - K'_{-\bar{k}} + i\eta T_{-\bar{k}})\tilde{\psi}$$
$$= (-S_{-\bar{k}} + i\eta(I + K'_{-\bar{k}})) \left(\sum_{l=1}^{N} r_l^2 (\hat{\varphi}(x_l') - y_l) \left. \frac{\partial \Phi_{-\bar{k}}(\cdot - x_l')}{\partial \nu} \right|_\Gamma \right), \tag{3.34}$$

$$(I - K_k - i\eta T_k)\tilde{\chi} \tag{3.35}$$
$$= (-S_k - i\eta(I + K_k')) \left[q_1^{-2} \left(\tilde{\psi} + \sum_{l=1}^{N} r_l^2 (y_l - \hat{\varphi}(x_l'))\, \Phi_{-\bar{k}}(\cdot - x_l')|_\Gamma \right) + h_0 \right],$$

$$\hat{\varphi}(x_i') = \mathcal{W}_k \tilde{\chi}(x_r') \tag{3.36}$$
$$-\mathcal{V}_k \left[q_1^{-2} \left(\tilde{\psi} + \sum_{l=1}^{N} r_l^2 (y_l - \hat{\varphi}(x_l'))\, \Phi_{-\bar{k}}(\cdot - x_l')|_\Gamma \right) + h_0 \right] \Bigg|_{x_i'}, \;\; i = \overline{1, N}.$$

in which η is an arbitrary real number such that $\eta \mathrm{Re}\, k > 0$. This system is uniquely solvable for all values of wave numbers k, $\mathrm{Im}\, k \geq 0$.

Remark 9. *Similar results can be obtained if the estimated functional has the form* (2.12).

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